

# DISTRIBUTIVE SEMILATTICES AS RETRACTS OF ULTRABOOLEAN ONES; FUNCTORIAL INVERSES WITHOUT ADJUNCTION

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*Souviens toi, ma petite Lynn,  
la perle du petit dragon...*

**ABSTRACT.** A  $\langle \vee, 0 \rangle$ -semilattice is *ultrabolean*, if it is a directed union of finite Boolean  $\langle \vee, 0 \rangle$ -semilattices. We prove that every distributive  $\langle \vee, 0 \rangle$ -semilattice is a retract of some ultrabolean  $\langle \vee, 0 \rangle$ -semilattice. This is established by proving that every finite distributive  $\langle \vee, 0 \rangle$ -semilattice is a retract of some finite Boolean  $\langle \vee, 0 \rangle$ -semilattice, and this in a *functorial* way. This result is, in turn, obtained as a particular case of a category-theoretical result that gives sufficient conditions, for a functor  $\Pi$ , to admit a right inverse. The particular functor  $\Pi$  used for the abovementioned result about ultrabolean semilattices has neither a right nor a left adjoint.

## 1. INTRODUCTION

Our general kind of problem is the following. We are given a functor  $\mathbf{F}$  from a category  $\mathcal{A}$  to a category  $\mathcal{B}$ , we wish to investigate whether  $\mathbf{F}$  has a right inverse (up to equivalence). Also, we suppose that we know how to do this on a subcategory of  $\mathcal{B}$ , or, more generally, on a given class of diagrams of  $\mathcal{B}$ . We wish to set a general framework that will enable us, under certain conditions, to find a right inverse of  $\mathbf{F}$  on a much larger class of diagrams of  $\mathcal{B}$ .

How to do this will be stated precisely in a further paper [17]. The present paper is intended to provide a start for that program, and it is motivated by the following example. We denote by  $\mathcal{L}$  the category of all lattices, by  $\mathcal{D}$  the category of all distributive  $\langle \vee, 0 \rangle$ -semilattices, and by  $\text{Con}_c: \mathcal{L} \rightarrow \mathcal{D}$  the functor that with a lattice  $L$  associates its semilattice  $\text{Con}_c L$  of compact congruences, extended naturally to lattice homomorphisms. It is a well-known open problem, stated by R. P. Dilworth in 1945, whether every distributive  $\langle \vee, 0 \rangle$ -semilattice is isomorphic to  $\text{Con}_c L$  for some lattice  $L$ . We wish to reduce that problem, or rather some stronger versions about diagrams of semilattices, to a smaller class of distributive  $\langle \vee, 0 \rangle$ -semilattices for which calculations are easier. Our candidate is the following.

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**Definition 1.1.** A  $\langle \vee, 0 \rangle$ -semilattice is *ultraboolean*, if it is a directed union of finite Boolean  $\langle \vee, 0 \rangle$ -semilattices.

Hence every ultraboolean  $\langle \vee, 0 \rangle$ -semilattice is distributive (the converse is trivially false, see the three-element chain).

For the present paper's needs, everything boils down to expressing members of the larger class (distributive semilattices) as *retracts* of the members of the smaller class (ultraboolean semilattices). Furthermore, such a retraction needs to be *functorial*. We shall refer to this problem as the *ultraboolean retraction problem*. At first sight, it is not clear whether the functoriality restriction might cause a problem. Indeed, every finite distributive lattice  $D$  is a retract of a finite Boolean lattice  $B$ . For example, as in [13, Section 1], we can embed  $D$  into the power set  $B = \mathfrak{P}(J(D))$ , where  $J(D)$  denotes the poset (i.e., partially ordered set) of join-irreducible elements of  $D$ , *via* the map

$$a \mapsto \{p \in J(D) \mid p \leq a\}.$$

This map has a retraction, given by  $X \mapsto \bigvee X$ . As in [13, Section 1], one can extend ‘canonically’ any  $\langle \vee, 0 \rangle$ -embedding  $f: D \hookrightarrow E$  to a  $\langle \vee, 0 \rangle$ -homomorphism  $g: \mathfrak{P}(J(D)) \rightarrow \mathfrak{P}(J(E))$ ; however, even for  $f = \text{id}_D$ , the map  $g$  might not be an embedding! Hence this ‘functor’ preserves neither monomorphisms nor, in fact, identities, and thus it is not sufficient to solve the ultraboolean retraction problem.

In order to solve that problem, we need to embed any finite distributive  $\langle \vee, 0 \rangle$ -semilattice  $D$  into some finite Boolean  $\langle \vee, 0 \rangle$ -semilattice  $\Phi(D)$ , *via* a  $\langle \vee, 0 \rangle$ -embedding  $\varepsilon_D: D \hookrightarrow \Phi(D)$ , with a retraction  $\mu_D: \Phi(D) \twoheadrightarrow D$ , these data being subjected to functoriality conditions, stated precisely in Section 3. Here are some caveats:

- Solving the problem ‘without the retraction’  $\mu_D$  is easy: namely, embed  $D$  into the universal Boolean semilattice  $\text{Bool}(D)$  over  $D$ . For this construction, the corresponding embedding  $\varepsilon_D: D \hookrightarrow \text{Bool}(D)$  is *not* a meet-embedding as a rule. An explicit construction is given by  $\text{Bool}(D) = \mathfrak{P}(D^=)$  (where  $D^= = D \setminus \{1\}$ ),  $\varepsilon_D(a) = \{x \in D^= \mid a \not\leq x\}$  (for all  $a \in D$ ). On the other hand, any  $\langle \vee, 0 \rangle$ -embedding  $f: D \hookrightarrow E$  is turned to a *lattice* embedding  $g: \text{Bool}(D) \hookrightarrow \text{Bool}(E)$ ! However, the retracts are lost, for the canonical retraction from  $\text{Bool}(D)$  onto  $D$  does not satisfy the required commutation conditions.
- For a finite distributive  $\langle \vee, 0 \rangle$ -semilattice  $D$ , the canonical map from  $D$  into  $\mathfrak{P}(J(D))$  is, in fact, a *lattice* embedding. However, the requirement that all the maps  $\varepsilon_D: D \hookrightarrow \Phi(D)$  be lattice embeddings is too strong to solve the ultraboolean retraction problem. This is showed by a counterexample in Section 10.

Nevertheless, we prove that the ultraboolean retraction problem has a positive solution. This result is, actually, an immediate application of a more general categorical principle, stated in Theorem 5.3. This principle states sufficient conditions for every object of a category  $\mathcal{A}$  to be a retract of some object of a category  $\mathcal{B}$ , and this functorially. Although some aspects of the formulations might remind of the Adjoint Functor Theorem, it is not hard to prove that in the particular case of the ultraboolean retraction problem, the functorial inverse that we construct does not arise from a functorial adjunction, see Proposition 9.6.

The importance of finite, simple, atomistic lattices for representation problems is highlighted in the paper of P. P. Pálffy and P. Pudlák [11], where it is proved that if a

finite, simple lattice  $L$  whose atoms join to the unit is isomorphic to the congruence lattice of a finite algebra, then it isomorphic to the congruence lattice of a finite set with a finite group action. With this in mind and by using a trick of G. Grätzer and E. T. Schmidt, we give, in Section 11, an easy proof of the result that every  $\langle \vee, 0 \rangle$ -semilattice is a retract of some directed  $\langle \vee, 0 \rangle$ -union of finite, (lattice-)simple, atomistic lattices, and this in a functorial way. Although this proof does not use the result of Theorem 5.3, further potential uses of Theorem 5.3 are suggested by open problems such as Problem 5 (see Section 12).

While the present paper deals with the *existence* of functorial retractions, the paper [17] deals with how to *use* functorial retractions in order to prove that certain functors have large range.

While this paper is mainly category-theoretical, it aims at building up tools that will be used later in universal algebra. For this reason, the author chose to write it in probably more detail than a category theorist would wish, with the hope to make it reasonably intelligible to members of both communities.

However, a direct semilattice-theoretical proof of Theorem 9.5 (solution of the ultraboolean retraction problem) is not easier than the categorical proof involving Theorem 5.3, and it does not lead itself to further potential generalizations such as those suggested in Section 12. This, together with the categorical approach required in [17], motivates our choice of the language of categories instead of the one of universal algebra.

## 2. BASIC CONCEPTS

Most of our categorical notions are borrowed from S. Mac Lane [10]. For a category  $\mathcal{C}$ , we shall denote by  $\text{Ob } \mathcal{C}$  the class of *objects* of  $\mathcal{C}$ , by  $\mathcal{C}^{\text{iso}}$  the category whose objects are those of  $\mathcal{C}$  and whose morphisms are the isomorphisms of  $\mathcal{C}$ . We shall denote by  $\text{dom } f$  the domain of a morphism  $f$  of  $\mathcal{C}$ . As usual, a morphism in  $\mathcal{C}$  is a *monic* (resp., a *section*), if it is left cancellable (resp., left invertible) for the composition of morphisms. Of course, every section is a monic.

We shall view every quasi-ordered set  $\langle P, \trianglelefteq \rangle$  as a category in which hom-sets have at most one element. Technically speaking, our quasi-ordered sets may be proper classes, but in our context this will create no difficulty. For  $p \trianglelefteq q$  in  $P$ , we shall denote by  $p \rightarrow q$  the unique morphism from  $p$  to  $q$ . An *ideal* of  $P$  is a subset  $X$  of  $P$  such that  $p \trianglelefteq x$  implies that  $p \in X$ , for all  $\langle p, x \rangle \in P \times X$ . We denote by  $\downarrow X$  the ideal generated by  $X$ , for all  $X \subseteq P$ , and we put  $\downarrow p = \downarrow \{p\}$ , for all  $p \in P$ . We put  $\mathbf{2} = \{0, 1\}$ , the two-element poset. For quasi-ordered sets  $\langle P, \trianglelefteq_P \rangle$  and  $\langle Q, \trianglelefteq_Q \rangle$ , a map  $f: P \rightarrow Q$  is an *embedding*, if  $x \trianglelefteq_P y$  iff  $f(x) \trianglelefteq_Q f(y)$ , for all  $x, y \in P$ ; we say that  $f$  is a *lower embedding*, if  $f$  is an embedding and the range of  $f$  is an ideal of  $Q$ .

For a meet-semilattice  $S$ , we put  $S^{\neq} = S \setminus \{1\}$  if  $S$  has a unit,  $S^{\neq} = S$  otherwise. Furthermore, we denote by  $M(S)$  the set of all meet-irreducible elements of  $S$ , that is, those  $u \in S^{\neq}$  such that  $u = x \wedge y$  implies that either  $u = x$  or  $u = y$ , for all  $x, y \in S$ . Dually, for a  $\langle \vee, 0 \rangle$ -semilattice  $S$ , we denote by  $J(S)$  the set of all join-irreducible elements of  $S$ .

We denote by  $\omega$  the set of all natural numbers and by  $\mathfrak{P}(X)$  the power set of  $X$ , for any set  $X$ .

### 3. FUNCTORIAL RETRACTS

**Definition 3.1.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be subcategories of a category  $\mathcal{C}$ . We denote by  $\text{Retr}(\mathcal{A}, \mathcal{B})$  the category whose objects and morphisms are the following:

- *Objects*: all quadruples  $\langle A, B, \varepsilon, \mu \rangle$ , where  $A \in \text{Ob } \mathcal{A}$ ,  $B \in \text{Ob } \mathcal{B}$ ,  $\varepsilon: A \rightarrow B$ ,  $\mu: B \rightarrow A$ , and  $\mu \circ \varepsilon = \text{id}_A$ .
- *Morphisms*: a morphism from  $\langle A, B, \varepsilon, \mu \rangle$  to  $\langle A', B', \varepsilon', \mu' \rangle$  is a pair  $\langle f, g \rangle$ , where  $f: A \rightarrow A'$  in  $\mathcal{A}$ ,  $g: B \rightarrow B'$  in  $\mathcal{B}$ ,  $g \circ \varepsilon = \varepsilon' \circ f$ , and  $\mu' \circ g = f \circ \mu$  (see Figure 3.1). Composition of morphisms is defined by the rule  $\langle f', g' \rangle \circ \langle f, g \rangle = \langle f' \circ f, g' \circ g \rangle$ .

In short,  $\text{Retr}(\mathcal{A}, \mathcal{B})$  is the category of all retractions of an object of  $\mathcal{B}$  onto an object of  $\mathcal{A}$ .

The *projection functor* from  $\text{Retr}(\mathcal{A}, \mathcal{B})$  to  $\mathcal{A}$  is the functor from  $\text{Retr}(\mathcal{A}, \mathcal{B})$  to  $\mathcal{A}$  that sends any object  $\langle A, B, \varepsilon, \mu \rangle$  to  $A$  and any morphism  $\langle f, g \rangle$  to  $f$ .

$$\begin{array}{ccc} B & \xrightarrow{g} & B' \\ \varepsilon \uparrow \mu & & \varepsilon' \uparrow \mu' \\ A & \xrightarrow{f} & A' \end{array}$$

FIGURE 3.1. Morphisms in  $\text{Retr}(\mathcal{A}, \mathcal{B})$ .

**Definition 3.2.** We say that  $\mathcal{A}$  is a *functorial retract* of  $\mathcal{B}$ , if the projection functor from  $\text{Retr}(\mathcal{A}, \mathcal{B})$  to  $\mathcal{A}$  has a right inverse. We shall call such an inverse a *functorial retraction* of  $\mathcal{A}$  to  $\mathcal{B}$ .

Hence a functorial retraction may be viewed as a triple  $\langle \Phi, \varepsilon, \mu \rangle$  that satisfies the following conditions:

- $\Phi$  is a functor from  $\mathcal{A}$  to  $\mathcal{B}$ .
- For every morphism  $f: X \rightarrow Y$  in  $\mathcal{A}$ , we have  $\varepsilon_X: X \rightarrow \Phi(X)$ ,  $\mu_X: \Phi(X) \rightarrow X$ ,  $\mu_X \circ \varepsilon_X = \text{id}_X$ ,  $\Phi(f) \circ \varepsilon_X = \varepsilon_Y \circ f$ , and  $\mu_Y \circ \Phi(f) = f \circ \mu_X$  (see Figure 3.2).

Observe that we do not require the diagram of Figure 3.2 to be commutative, for example,  $\Phi(f) \neq \varepsilon_Y \circ f \circ \mu_X$  in general.

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{\Phi(f)} & \Phi(Y) \\ \varepsilon_X \uparrow \mu_X & & \varepsilon_Y \uparrow \mu_Y \\ X & \xrightarrow{f} & Y \end{array}$$

FIGURE 3.2. Functorial retraction of  $\mathcal{A}$  to  $\mathcal{B}$ .

## 4. SHELTERING BETWEEN FULL SUBCATEGORIES

**Definition 4.1.** An *ideal of monics* of a category  $\mathcal{C}$  is a subcategory  $\mathcal{M}$  of  $\mathcal{C}$  satisfying the following conditions:

- (i) Every identity of  $\mathcal{C}$  belongs to  $\mathcal{M}$ .
- (ii)  $g \circ f \in \mathcal{M}$  implies that  $f \in \mathcal{M}$ , for all morphisms  $f$  and  $g$  of  $\mathcal{C}$  such that  $g \circ f$  is defined.
- (iii) Every morphism in  $\mathcal{M}$  is a monic.

Of course, the monics of  $\mathcal{C}$  form the largest ideal of monics of  $\mathcal{C}$ , while the sections of  $\mathcal{C}$  form the smallest ideal of monics of  $\mathcal{C}$ . An example of often used ideal of monics distinct from both the class of all monics and the class of all sections is constructed within the category of all commutative monoids, as the ideal of all one-to-one monoid homomorphisms  $f$  that satisfy  $f(x) \leq f(y) \Rightarrow x \leq y$ , where  $x \leq y$  is an abbreviation for  $(\exists z)(x + z = y)$ .

**Definition 4.2.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be full subcategories of a category  $\mathcal{C}$  and let  $\mathcal{M}$  be an ideal of monics of  $\mathcal{C}$ . A *shelter of  $\mathcal{C}$  by  $\mathcal{B}$  with respect to  $\langle \mathcal{A}, \mathcal{M} \rangle$*  consists of the following data (illustrated on Figure 4.1):

- (i) A functor  $\mathbf{B}$  from  $\mathcal{C}^{\text{iso}}$  to  $\mathcal{B}^{\text{iso}}$ .
- (ii) A natural transformation  $S \mapsto \eta_S$  from the identity functor on  $\mathcal{C}^{\text{iso}}$  to the functor  $\mathbf{B}$ , such that  $\eta_S \in \mathcal{M}$ , for every  $S \in \text{Ob } \mathcal{C}$ .
- (iii) A map that with every morphism  $g: S \rightarrow A$ , where  $S \in \text{Ob } \mathcal{C}$  and  $A \in \text{Ob } \mathcal{A} \cup \text{Ob } \mathcal{B}$ , associates a morphism  $g^{\mathbf{B}}: \mathbf{B}(S) \rightarrow A$  such that  $g = g^{\mathbf{B}} \circ \eta_S$ .

Furthermore, we require the following conditions to be satisfied:

- (1) For every isomorphism  $f: S \rightarrow T$  in  $\mathcal{C}$  and every  $g: T \rightarrow A$ , with  $A \in \text{Ob } \mathcal{A} \cup \text{Ob } \mathcal{B}$ ,  $g^{\mathbf{B}} \circ \mathbf{B}(f) = (g \circ f)^{\mathbf{B}}$  (see Figure 4.2(1)).
- (2) For every  $h: S \rightarrow A$  and every isomorphism  $u: A \rightarrow A'$  with either  $A, A' \in \text{Ob } \mathcal{A}$  or  $A, A' \in \text{Ob } \mathcal{B}$ ,  $(u \circ h)^{\mathbf{B}} = u \circ h^{\mathbf{B}}$  (see Figure 4.2(2)).

The figure consists of three separate commutative diagrams:

- The first diagram shows  $S$  at the bottom and  $\mathbf{B}(S)$  at the top. An arrow labeled  $\eta_S \in \mathcal{M}$  goes from  $S$  to  $\mathbf{B}(S)$ .
- The second diagram shows  $S$  at the bottom,  $T$  in the middle, and  $\mathbf{B}(T)$  at the top. There are two arrows from  $S$  to  $T$  labeled  $f$  and  $\eta_S$ , and two arrows from  $T$  to  $\mathbf{B}(T)$  labeled  $\eta_T$  and  $\mathbf{B}(f)$ . The top row is marked with  $\cong$  above the arrows.
- The third diagram shows  $S$  at the bottom,  $\mathbf{B}(S)$  at the top, and  $A$  on the right. There are two arrows from  $S$  to  $A$  labeled  $g$  and  $g^{\mathbf{B}}$ , and an arrow from  $\mathbf{B}(S)$  to  $A$  labeled  $\eta_S$ .

FIGURE 4.1. Data describing a shelter.

**Remark 4.3.** In all examples considered in this paper,  $\mathcal{B}$  is contained in  $\mathcal{A}$ . One can then say that a shelter is a weak reflection of  $\mathcal{C}$  to  $\mathcal{A}$  which is everywhere a monic (i.e., all arrows  $\eta_S$  are monics), has values in  $\mathcal{B}$  (in case  $\mathcal{B} \subseteq \mathcal{A}$ ), and is functorial on isomorphisms.

## 5. STATEMENT OF THE MAIN THEOREM

**Definition 5.1.** Let  $\mathcal{M}$  be an ideal of monics of a category  $\mathcal{C}$ . For  $S \in \text{Ob } \mathcal{C}$ , we denote by  $\mathcal{M}(S)$  the set of all morphisms  $u: X \rightarrow S$  in  $\mathcal{M}$ , and we put

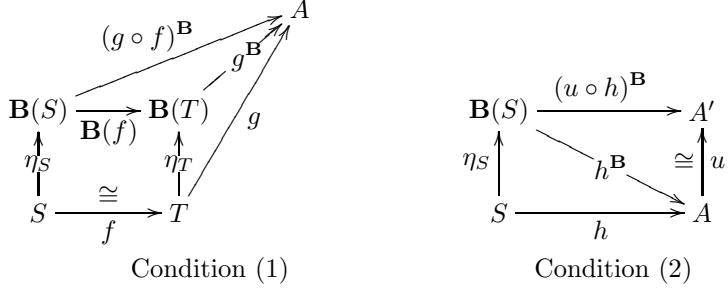


FIGURE 4.2. Additional features of a shelter.

$\mathcal{M}^*(S) = \mathcal{M}(S) \setminus \mathcal{M}^{\text{iso}}$ . Furthermore, for  $u: X \rightarrow S$  and  $v: Y \rightarrow S$  in  $\mathcal{M}$ , we put

$$u \trianglelefteq_S v \iff (\exists f: X \rightarrow Y)(u = v \circ f); \quad (5.1)$$

$$u \sim_S v \iff (u \trianglelefteq_S v \text{ and } v \trianglelefteq_S u); \quad (5.2)$$

$$u \triangleleft_S v \iff (u \trianglelefteq_S v \text{ and } v \not\trianglelefteq_S u). \quad (5.3)$$

Obviously,  $\trianglelefteq_S$  is a quasi-ordering on  $\mathcal{M}(S)$  and  $\sim_S$  is the associated equivalence. In case  $u \trianglelefteq_S v$ , we shall denote by  $u/v$  the unique  $f: X \rightarrow Y$  satisfying  $u = v \circ f$ . Necessarily,  $f \in \mathcal{M}$ , and  $f$  is an isomorphism iff  $u \sim_S v$ . We shall denote by  $\text{lh } S$  (the *length* of  $S$ ) the length of the quasi-ordered set  $(\mathcal{M}(S), \trianglelefteq_S)$  in case  $\mathcal{M}(S)$  has finite length. The blocks of  $\sim_S$  will be called the  $\mathcal{M}$ -*subobjects* of  $S$ .

**Lemma 5.2.** *Let  $\mathcal{M}$  be an ideal of monics of a category  $\mathcal{C}$  and let  $f: X \rightarrow Y$  in  $\mathcal{M}$ . Then the map  $\mathcal{M}(f): \mathcal{M}(X) \rightarrow \mathcal{M}(Y)$ ,  $u \mapsto f \circ u$  is a lower embedding. Furthermore, if both  $X$  and  $Y$  have finite length, then  $f$  is an isomorphism iff  $\text{lh } X = \text{lh } Y$ .*

*Proof.* Verifying that  $\mathcal{M}(f)$  is a lower embedding is a straightforward exercise. If  $f$  is an isomorphism, then so is  $\mathcal{M}(f)$ , thus  $\text{lh } X = \text{lh } Y$ . If  $f$  is not an isomorphism, then  $\text{lh } X = \text{height}_{\mathcal{M}(X)}(\text{id}_X) = \text{height}_{\mathcal{M}(Y)}(f) < \text{height}_{\mathcal{M}(Y)}(\text{id}_Y) = \text{lh } Y$ .  $\square$

Now we state the main technical result of the paper.

**Theorem 5.3.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be full subcategories of a category  $\mathcal{C}$  and let  $\mathcal{M}$  be an ideal of monics of  $\mathcal{C}$ . We assume the following:*

- (i) *Every diagram of  $\mathcal{C}$ , indexed by a finite poset, and with vertices either in  $\mathcal{A}$  or in  $\mathcal{B}$ , has a colimit.*
- (ii) *Every object of  $\mathcal{A}$  has only finitely many  $(\mathcal{A} \cap \mathcal{M})$ -subobjects.*
- (iii)  *$\mathcal{C}$  is sheltered by  $\mathcal{B}$  with respect to  $(\mathcal{A}, \mathcal{M})$ .*
- (iv) *For every  $\varphi: A_0 \rightarrow A_1$  in  $\mathcal{M}$  and every section  $\varepsilon_0: A_0 \rightarrow B_0$ , with  $A_0, A_1 \in \text{Ob } \mathcal{A}$  and  $B_0 \in \text{Ob } \mathcal{B}$ , there is  $S \in \text{Ob } \mathcal{C}$ , together with  $\varepsilon_1: A_1 \rightarrow S$  and  $\psi: B_0 \rightarrow S$  both in  $\mathcal{M}$ , such that  $\psi \circ \varepsilon_0 = \varepsilon_1 \circ \varphi$ .*

*Then  $\mathcal{A} \cap \mathcal{M}$  is a functorial retract of  $\mathcal{B} \cap \mathcal{M}$ .*

From now on until the end of Section 8, we shall assume that  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ , and  $\mathcal{M}$  satisfy the assumptions of Theorem 5.3, with a shelter  $\mathbf{B}$  denoted as in Section 4. The functorial retraction of Theorem 5.3 will be constructed explicitly, in terms of categorical operations and  $\mathbf{B}$ .

Our next lemma states that in item (iv) of Theorem 5.3, we may assume that  $S \in \text{Ob } \mathcal{B}$ .

**Lemma 5.4.** *For all  $\varphi: A_0 \rightarrow A_1$  in  $\mathcal{M}$  and every section  $\varepsilon_0: A_0 \rightarrow B_0$ , with  $A_0, A_1 \in \text{Ob } \mathcal{A}$  and  $B_0 \in \text{Ob } \mathcal{B}$ , there is  $B \in \text{Ob } \mathcal{B}$ , together with  $\varepsilon_1: A_1 \rightarrow B$  and  $\psi: B_0 \rightarrow B$  both in  $\mathcal{M}$ , such that  $\psi \circ \varepsilon_0 = \varepsilon_1 \circ \varphi$ .*

*Proof.* Consider  $S$ ,  $\varepsilon_1$ , and  $\psi$  obtained from (iv) of Theorem 5.3. Replace  $S$  by  $B = \mathbf{B}(S)$ ,  $\varepsilon_1$  by  $\eta_S \circ \varepsilon_1$ , and  $\psi$  by  $\eta_S \circ \psi$ .  $\square$

*Remark 5.5.* Our formulation of Theorem 5.3 is a compromise between conciseness and generality. As one can never be sure about future applications, let us mention a few possible weakenings of its assumptions. Assumption (ii) can be weakened, by putting a cardinal upper bound, say,  $\kappa$ , on the number of subobjects of all objects of  $\mathcal{A}$ . Then, Assumption (i) needs to be extended to diagrams indexed by posets of size below  $\kappa$ . However, in order to be able to define the (ordinal) length, and, in particular, to get an analogue of Lemma 5.2, we need to keep the assumption that each poset  $\langle \mathcal{M}(S), \leq_S \rangle$  is *well-founded* (the terminology *artinian* is also used), that is, every nonempty subset has a minimal element. Finally, the diagrams involved in Assumption (i) are fairly special, for example, they have at least one vertex in  $\mathcal{A}$  and all their arrows in  $\mathcal{M}$ . However, we know no situation where such generalizations would be of any practical use.

## 6. INDUCTIVE CONSTRUCTION OF $\Phi$ , $\varepsilon$ , $\mu$

Denote by  $\mathcal{A}_n$  the full subcategory of  $\mathcal{A}$  whose objects are those  $X \in \text{Ob } \mathcal{A}$  such that  $\text{lh } X < n$ , for every natural number  $n$ . Of course,  $\mathcal{A}_0$  is the empty category.

Fix a natural number  $n$ , and suppose having constructed a functor  $\Phi$  from  $\mathcal{A}_n \cap \mathcal{M}$  to  $\mathcal{B}$ , together with a system of morphisms  $\varepsilon_X: X \rightarrow \Phi(X)$  and  $\mu_X: \Phi(X) \rightarrow X$ , for  $X \in \text{Ob } \mathcal{A}_n$ , such that the following induction hypothesis is satisfied:

$$\mu_X \circ \varepsilon_X = \text{id}_X, \quad \Phi(f) \circ \varepsilon_X = \varepsilon_Y \circ f, \quad \text{and} \quad \mu_Y \circ \Phi(f) = f \circ \mu_X, \quad (6.1)$$

for every morphism  $f: X \rightarrow Y$  in  $\mathcal{A}_n \cap \mathcal{M}$ .

We do not assume, for the moment, that  $\Phi$  sends morphisms in  $\mathcal{M}$  to morphisms in  $\mathcal{M}$ . So, for  $f: X \rightarrow Y$  in  $\mathcal{A}_n \cap \mathcal{M}$ , all we know is that  $\Phi(f): \Phi(X) \rightarrow \Phi(Y)$  in  $\mathcal{B}$ . We fix an object  $A$  of  $\mathcal{A}$  such that  $\text{lh } A = n$ .

Let us outline the construction. We shall introduce a diagram  $\rho_A$ , indexed by a quasi-ordered set  $\langle \tilde{\mathcal{A}} \upharpoonright A, \leq_A \rangle$ . Intuitively,  $\rho_A$  consists of all spans  $\langle u, \eta_X \rangle$ , where  $X \in \text{Ob } \mathcal{A}$ ,  $u: X \rightarrow A$  in  $\mathcal{M}$ , and  $\eta_X$  is either  $\varepsilon_X$ , in case  $u \notin \mathcal{M}^{\text{iso}}$ , or  $\text{id}_X$ , in case  $u \in \mathcal{M}^{\text{iso}}$ . We equip these objects with the obvious arrows, see (6.2). An important auxiliary construction is the colimit of  $\rho_A$ , which consists of an object  $\Phi_*(A)$ , together with arrows  $\varepsilon^A: A \rightarrow \Phi_*(A)$  and  $\Phi_*(u): \Phi(X) \rightarrow \Phi_*(A)$ , for  $u: X \rightarrow A$  in  $\mathcal{M} \setminus \mathcal{M}^{\text{iso}}$ , subjected to the commutation relations illustrated on Figure 6.1. The resulting natural transformation  $A \mapsto \varepsilon^A$  from the identity to  $\Phi_*$  is split by  $\mu^-: \Phi_* \rightarrow \text{id}$ ,  $A \mapsto \mu^A$ , living in  $\mathcal{C}$  and given as follows:  $\mu^A$  is induced by the cocone  $\langle \text{id}_A, \langle u \circ \mu_X \mid u: X \rightarrow A \text{ in } \mathcal{M} \setminus \mathcal{M}^{\text{iso}} \rangle \rangle$ , see Lemma 6.3. We observe that the definition of  $\Phi_*(A)$  does not use only all previous values of  $\Phi_*$ , but really all previous values of  $\Phi$ .

The shelter  $\mathbf{B}$  is used in order to define  $\Phi(A)$ : namely,  $\Phi(A) = \mathbf{B}(\Phi_*(A))$ , see (6.4). The natural transformation  $A \mapsto \varepsilon_A$ , its section  $A \mapsto \mu_A$ , and the

arrows  $\Phi(u)$ , for  $u: X \rightarrow A$  in  $\mathcal{M} \setminus \mathcal{M}^{\text{iso}}$ , are then defined in the natural way, see (6.5)–(6.7). The rest of the section is then devoted to proving that this extension of  $\Phi$  on  $\mathcal{A}_{n+1}$  can, indeed, be further extended to a functor. Although it will turn out that  $\Phi$  preserves  $\mathcal{M}$ , this is a nontrivial fact and it will not be assumed as an induction hypothesis through the construction. We shall establish this fact in Sections 7 and 8.

Now let us go to the details. We put  $\mathcal{A} \upharpoonright A = (\mathcal{A} \cap \mathcal{M})(A)$  and  $\mathcal{A} \upharpoonright^* A = (\mathcal{A} \upharpoonright A) \setminus \mathcal{M}^{\text{iso}}$ . For  $u: X \rightarrow A$  and  $v: Y \rightarrow A$  in  $\mathcal{A} \upharpoonright A$  with  $u \trianglelefteq_A v$ , we shall often identify the morphism (in  $\mathcal{A} \upharpoonright A$ )  $u \rightarrow v$  with  $f = u/v$ , which is a morphism (in  $\mathcal{A} \cap \mathcal{M}$ ) from  $X$  to  $Y$ . We endow the set

$$\tilde{\mathcal{A}} \upharpoonright A = \{ \langle u, i \rangle \in (\mathcal{A} \upharpoonright A) \times \mathbf{2} \mid i = 1 \Rightarrow u \notin \mathcal{M}^{\text{iso}} \}$$

with the partial quasi-ordering, that we shall still denote by  $\trianglelefteq_A$ , defined by

$$\langle u, i \rangle \trianglelefteq_A \langle v, j \rangle \iff (u \trianglelefteq_A v \text{ and } i \leq j), \quad \text{for all } \langle u, i \rangle, \langle v, j \rangle \in \tilde{\mathcal{A}} \upharpoonright A.$$

For  $\langle u, i \rangle \in \text{Ob}(\tilde{\mathcal{A}} \upharpoonright A)$ , where  $u: X \rightarrow A$ , we define

$$\rho_A(\langle u, i \rangle) = \begin{cases} X, & \text{if } i = 0, \\ \Phi(X), & \text{if } i = 1. \end{cases}$$

(Observe that  $i = 1$  implies that  $u \notin \mathcal{M}^{\text{iso}}$ , thus  $\text{lh } X < \text{lh } A$ , and thus  $\Phi(X)$  is defined.) Furthermore, if  $\langle u, i \rangle \trianglelefteq_A \langle v, j \rangle$  in  $\tilde{\mathcal{A}} \upharpoonright A$ , we put

$$\rho_A(\langle u, i \rangle \rightarrow \langle v, j \rangle) = \begin{cases} u/v, & \text{if } i = j = 0, \\ \Phi(u/v) \circ \varepsilon_X, & \text{if } i = 0 \text{ and } j = 1, \\ \Phi(u/v), & \text{if } i = j = 1. \end{cases} \quad (6.2)$$

**Lemma 6.1.** *The correspondence  $\rho_A$  defines a functor from  $\tilde{\mathcal{A}} \upharpoonright A$  to  $\mathcal{M}$ .*

*Proof.* It is obvious that  $\rho_A(\langle u, i \rangle \rightarrow \langle v, j \rangle)$  is a morphism from  $\rho_A(\langle u, i \rangle)$  to  $\rho_A(\langle v, j \rangle)$ , and that  $\rho_A$  sends identities to identities. Now let  $\langle u, i \rangle \trianglelefteq_A \langle v, j \rangle \trianglelefteq_A \langle w, k \rangle$  in  $\tilde{\mathcal{A}} \upharpoonright A$ , we need to verify the equality

$$\rho_A(\langle u, i \rangle \rightarrow \langle w, k \rangle) = \rho_A(\langle v, j \rangle \rightarrow \langle w, k \rangle) \circ \rho_A(\langle u, i \rangle \rightarrow \langle v, j \rangle). \quad (6.3)$$

Let  $u: X \rightarrow A$ ,  $v: Y \rightarrow A$ , and  $w: Z \rightarrow A$ , put  $f = u/v$  and  $g = v/w$ . We separate cases.

**Case 1.**  $i = j = k = 0$ . Then

$$\rho_A(\langle v, j \rangle \rightarrow \langle w, k \rangle) \circ \rho_A(\langle u, i \rangle \rightarrow \langle v, j \rangle) = g \circ f = \rho_A(\langle u, i \rangle \rightarrow \langle w, k \rangle).$$

**Case 2.**  $i = j = 0$ ,  $k = 1$ . Then

$$\begin{aligned} \rho_A(\langle v, j \rangle \rightarrow \langle w, k \rangle) \circ \rho_A(\langle u, i \rangle \rightarrow \langle v, j \rangle) &= \Phi(g) \circ \varepsilon_Y \circ f \\ &= \Phi(g) \circ \Phi(f) \circ \varepsilon_X \\ &= \Phi(g \circ f) \circ \varepsilon_X \\ &= \rho_A(\langle u, i \rangle \rightarrow \langle w, k \rangle). \end{aligned}$$

**Case 3.**  $i = 0$ ,  $j = k = 1$ . Then

$$\begin{aligned} \rho_A(\langle v, j \rangle \rightarrow \langle w, k \rangle) \circ \rho_A(\langle u, i \rangle \rightarrow \langle v, j \rangle) &= \Phi(g) \circ \Phi(f) \circ \varepsilon_X \\ &= \Phi(g \circ f) \circ \varepsilon_X \\ &= \rho_A(\langle u, i \rangle \rightarrow \langle w, k \rangle). \end{aligned}$$

**Case 4.**  $i = j = k = 1$ . Then

$$\rho_A(\langle v, j \rangle \rightarrow \langle w, k \rangle) \circ \rho_A(\langle u, i \rangle \rightarrow \langle v, j \rangle) = \Phi(g) \circ \Phi(f) = \Phi(g \circ f) = \rho_A(\langle u, i \rangle \rightarrow \langle w, k \rangle).$$

This concludes the proof.  $\square$

**Lemma 6.2.** *The functor  $\rho_A$  has a colimit in  $\mathcal{C}$ .*

*Proof.* It follows from Assumption (ii) of Theorem 5.3 that  $\mathcal{A} \upharpoonright A$  is equivalent to a finite poset; hence  $\tilde{\mathcal{A}} \upharpoonright A$  is also equivalent to a finite poset. Since the colimit is a categorical concept, the conclusion follows from Assumption (i) of Theorem 5.3.  $\square$

A colimit of  $\rho_A$  is given by an object  $\Phi_*(A)$ , together with a system of morphisms  $\theta_{\langle u, i \rangle}: \rho_A(\langle u, i \rangle) \rightarrow \Phi_*(A)$ , for all  $\langle u, i \rangle \in \tilde{\mathcal{A}} \upharpoonright A$ , subjected to certain commutation relations. In case  $u \in \mathcal{A} \upharpoonright^* A$ , the equality  $\theta_{\langle u, 0 \rangle} = \theta_{\langle u, 1 \rangle} \circ \varepsilon_{\text{dom } u}$  holds. Hence, putting  $\Phi_*(u) = \theta_{\langle u, 1 \rangle}$  and  $\varepsilon^A = \theta_{\langle \text{id}_A, 0 \rangle}$ , we obtain that the colimit of  $\rho_A$  is given by the object  $\Phi_*(A)$ , together with morphisms  $\Phi_*(u): \Phi(\text{dom } u) \rightarrow \Phi_*(A)$ , for  $u \in \mathcal{A} \upharpoonright^* A$ , and  $\varepsilon^A: A \rightarrow \Phi_*(A)$ , subjected to the commutativity of the diagrams represented on Figure 6.1 and the universality of  $\Phi_*(A)$  together with the system of morphisms consisting of all  $\Phi_*(u)$ -s and  $\varepsilon^A$ . Observe that for  $n = 0$ , this reduces to the universality of  $\varepsilon^A: A \rightarrow \Phi_*(A)$ ; so, in that case, we may take  $\Phi_*(A) = A$  and  $\varepsilon^A = \text{id}_A$ .

FIGURE 6.1. The colimit of  $\rho_A$ .

**Lemma 6.3.** *There exists a unique morphism  $\mu^A: \Phi_*(A) \rightarrow A$  such that  $\mu^A \circ \Phi_*(u) = u \circ \mu_{\text{dom } u}$ , for every  $u \in \mathcal{A} \upharpoonright^* A$ , and  $\mu^A \circ \varepsilon^A = \text{id}_A$ .*

*Proof.* We put  $\tau_u = u \circ \mu_{\text{dom } u}$ , for all  $u \in \mathcal{A} \upharpoonright^* A$ . By the universality of the colimit, it suffices to verify that the diagrams of Figure 6.2 commute, for all  $u: X \rightarrow A$  and  $v: Y \rightarrow A$  in  $\mathcal{A} \upharpoonright^* A$  and  $f = u/v$ .

FIGURE 6.2. Retracting  $\Phi_*(A)$  onto  $A$ .

Left hand side diagram:  $\tau_v \circ \Phi(f) = v \circ \mu_Y \circ \Phi(f) = v \circ f \circ \mu_X = u \circ \mu_X = \tau_u$ .  
Right hand side diagram:  $\tau_u \circ \varepsilon_X = u \circ \mu_X \circ \varepsilon_X = u$ . This concludes the proof.  $\square$

Now we are ready to define  $\Phi(A)$ ,  $\varepsilon_A$ ,  $\mu_A$ , and  $\Phi(u)$ , for  $u: X \rightarrow A$  in  $\mathcal{A} \upharpoonright^* A$ :

$$\Phi(A) = \mathbf{B}(\Phi_*(A)); \quad (6.4)$$

$$\varepsilon_A = \eta_{\Phi_*(A)} \circ \varepsilon^A; \quad (6.5)$$

$$\mu_A = (\mu^A)^{\mathbf{B}}. \quad (6.6)$$

$$\Phi(u) = \eta_{\Phi_*(A)} \circ \Phi_*(u). \quad (6.7)$$

These maps are represented on Figure 6.3. They satisfy the relations  $\mu^A \circ \varepsilon^A = \mu_A \circ \varepsilon_A = \text{id}_A$ ,  $\varepsilon_A = \eta_{\Phi_*(A)} \circ \varepsilon^A$ ,  $\mu^A = \mu_A \circ \eta_{\Phi_*(A)}$ , and  $\Phi(u) = \eta_{\Phi_*(A)} \circ \Phi_*(u)$ .

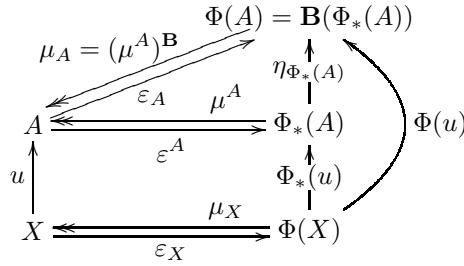


FIGURE 6.3. Defining  $\Phi(A)$ ,  $\varepsilon_A$ ,  $\mu_A$ , and  $\Phi(u)$ .

The computations of the relations  $\Phi(u) \circ \varepsilon_X = \varepsilon_A \circ u$  and  $\mu_A \circ \Phi(u) = u \circ \mu_X$  can be followed on Figures 6.1 and 6.3:

$$\begin{aligned} \Phi(u) \circ \varepsilon_X &= \eta_{\Phi_*(A)} \circ \Phi_*(u) \circ \varepsilon_X = \eta_{\Phi_*(A)} \circ \varepsilon^A \circ u = \varepsilon_A \circ u, \\ \mu_A \circ \Phi(u) &= \mu_A \circ \eta_{\Phi_*(A)} \circ \Phi_*(u) = \mu^A \circ \Phi_*(u) = u \circ \mu_X. \end{aligned}$$

In order to complete the extension of  $\Phi$  to all morphisms, it remains to define  $\Phi(g)$ , where  $g: A \rightarrow A'$  in  $\mathcal{A} \cap \mathcal{M}$  and  $\text{lh } A = \text{lh } A' = n$ . Observe that, by Lemma 5.2,  $g$  is an isomorphism. Moreover, if  $u: X \rightarrow A$  and  $v: Y \rightarrow A'$  belong to  $\mathcal{A} \upharpoonright^* A$  with  $u \trianglelefteq_A v$  and putting  $f = u/v$ , the diagrams of Figure 6.4 commute.

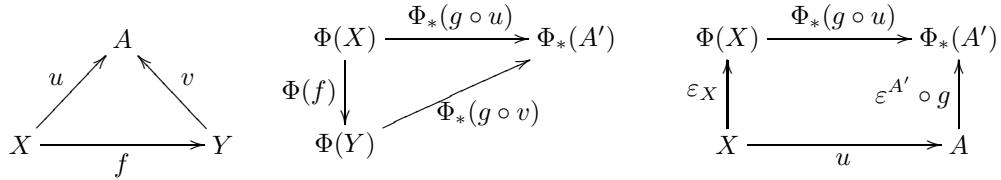


FIGURE 6.4. Putting  $\Phi_*(A')$  above the diagram defining  $\Phi_*(A)$ .

Therefore, by the universal property of  $\Phi_*(A)$  and the associated limiting morphisms, there exists a unique morphism  $\bar{g}: \Phi_*(A) \rightarrow \Phi_*(A')$  such that

$$\bar{g} \circ \varepsilon^A = \varepsilon^{A'} \circ g \text{ and } \Phi_*(g \circ u) = \bar{g} \circ \Phi_*(u), \text{ for all } u \in \mathcal{A} \upharpoonright^* A. \quad (6.8)$$

Symmetrically, there exists a unique morphism  $\bar{g}' : \Phi_*(A') \rightarrow \Phi_*(A)$  such that

$$\bar{g}' \circ \varepsilon^{A'} = \varepsilon^A \circ g^{-1} \text{ and } \Phi_*(u) = \bar{g}' \circ \Phi_*(g \circ u), \text{ for all } u \in \mathcal{A} \upharpoonright^* A. \quad (6.9)$$

Again by using the universal property defining  $\Phi_*(A)$  and  $\Phi_*(A')$ , we obtain that  $\bar{g}$  and  $\bar{g}'$  are mutually inverse isomorphisms. We define

$$\Phi_*(g) = \bar{g} \text{ and } \Phi(g) = \mathbf{B}(\Phi_*(g)). \quad (6.10)$$

So  $\Phi_*(g)$  is an isomorphism from  $\Phi_*(A)$  onto  $\Phi_*(A')$ , and, since  $\mathbf{B}$  is a functor from  $\mathcal{C}^{\text{iso}}$  to  $\mathcal{B}^{\text{iso}}$ ,  $\Phi(g)$  is an isomorphism from  $\Phi(A)$  onto  $\Phi(A')$ .

**Lemma 6.4.**  $\mu^{A'} \circ \Phi_*(g) = g \circ \mu^A$ .

*Proof.* By using the universal property defining  $\Phi_*(A)$ , it suffices to verify that the diagram represented on Figure 6.5 commutes, in both cases  $h = \mu^{A'} \circ \Phi_*(g \circ u)$  and  $h = g \circ \mu^A$ , for all  $u: X \rightarrow A$  in  $\mathcal{A} \upharpoonright A$ . Of course, none of the arrows of Figure 6.5 except  $\varepsilon^A$ ,  $g$ , and  $h$  are needed in case  $n = 0$ , in which case  $\Phi_*(A) = A$  and  $\varepsilon^A = \text{id}_A$ . The details of the computations use (6.8), (6.10), and Lemma 6.3;

$$\begin{array}{ccccc}
& & \mu^{A'} \circ \Phi_*(g \circ u) & \longrightarrow & A' \\
& \swarrow & & \nearrow h & \\
\Phi(X) & \xrightarrow{\Phi_*(u)} & \Phi_*(A) & \xrightarrow{g} & A' \\
\downarrow \varepsilon_X & & \downarrow \varepsilon^A & & \downarrow g \\
X & \xrightarrow{u} & A & \xrightarrow{h} & A'
\end{array}$$

FIGURE 6.5. Characterizing a morphism from  $\Phi_*(A)$  to  $A'$ .

they are as follows:

$$\begin{aligned}
\mu^{A'} \circ \Phi_*(g) \circ \Phi_*(u) &= \mu^{A'} \circ \Phi_*(g \circ u), \\
g \circ \mu^A \circ \Phi_*(u) &= g \circ u \circ \mu_X = \mu^{A'} \circ \Phi_*(g \circ u), \\
\mu^{A'} \circ \Phi_*(g) \circ \varepsilon^A &= \mu^{A'} \circ \varepsilon^{A'} \circ g = g, \\
g \circ \mu^A \circ \varepsilon^A &= g.
\end{aligned}$$

This completes the proof.  $\square$

**Lemma 6.5.**  $\Phi(g) \circ \varepsilon_A = \varepsilon_{A'} \circ g$  and  $\mu_{A'} \circ \Phi(g) = g \circ \mu_A$ .

*Proof.* By using (6.5), (6.8), and (6.10), we obtain

$$\Phi(g) \circ \varepsilon_A = \Phi(g) \circ \eta_{\Phi_*(A)} \circ \varepsilon^A = \eta_{\Phi_*(A')} \circ \Phi_*(g) \circ \varepsilon^A = \eta_{\Phi_*(A')} \circ \varepsilon^{A'} \circ g = \varepsilon_{A'} \circ g.$$

Furthermore, since  $\mathbf{B}$  is a shelter, we obtain, by using (6.6), the following equalities:

$$\begin{aligned}
\mu_{A'} \circ \Phi(g) &= (\mu^{A'})^{\mathbf{B}} \circ \mathbf{B}(\Phi_*(g)) = (\mu^{A'} \circ \Phi_*(g))^{\mathbf{B}}, \\
g \circ \mu_A &= g \circ (\mu^A)^{\mathbf{B}} = (g \circ \mu^A)^{\mathbf{B}}.
\end{aligned}$$

Therefore, by Lemma 6.4,  $\mu_{A'} \circ \Phi(g) = g \circ \mu_A$ .  $\square$

At this stage, we have extended  $\Phi$  to  $\mathcal{A}_{n+1} \cap \mathcal{M}$ , up to verification of preservation of composition by  $\Phi$ . Proving this preservation is the object of the next three lemmas.

**Lemma 6.6.** *Let  $A_0, A_1, A_2 \in \text{Ob } \mathcal{A}$  with  $\text{lh } A_0 = \text{lh } A_1 = \text{lh } A_2 = n$ , let  $f: A_0 \rightarrow A_1$  and  $g: A_1 \rightarrow A_2$  in  $\mathcal{M}$ . Then  $\Phi_*(g \circ f) = \Phi_*(g) \circ \Phi_*(f)$  and  $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$ .*

*Proof.* By Lemma 5.2, both  $f$  and  $g$  are isomorphisms. By using (6.8) and (6.10), we obtain

$$\Phi_*(g) \circ \Phi_*(f) \circ \varepsilon^{A_0} = \Phi_*(g) \circ \varepsilon^{A_1} \circ f = \varepsilon^{A_2} \circ g \circ f,$$

and, for all  $u \in \mathcal{A} \restriction^* A_0$ ,

$$\Phi_*(g \circ f \circ u) = \Phi_*(g) \circ \Phi_*(f \circ u) = \Phi_*(g) \circ \Phi_*(f) \circ \Phi_*(u).$$

Since these properties determine  $\Phi_*(g \circ f)$ , we obtain that  $\Phi_*(g \circ f) = \Phi_*(g) \circ \Phi_*(f)$ . Since  $\mathbf{B}$  is a functor from  $\mathcal{C}^{\text{iso}}$  to  $\mathcal{B}^{\text{iso}}$ , the equality  $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$  follows.  $\square$

**Lemma 6.7.** *Let  $X \in \text{Ob } \mathcal{A}_n$  and let  $A, A' \in \text{Ob } \mathcal{A}$  such that  $\text{lh } A = \text{lh } A' = n$ , let  $u: X \rightarrow A$  and let  $g: A \rightarrow A'$  in  $\mathcal{M}$ . Then  $\Phi(g \circ u) = \Phi(g) \circ \Phi(u)$ .*

*Proof.* By Lemma 5.2,  $g$  is an isomorphism. By using (6.7), (6.8), and (6.10), we obtain

$$\begin{aligned} \Phi(g \circ u) &= \eta_{\Phi_*(A')} \circ \Phi_*(g \circ u) = \eta_{\Phi_*(A')} \circ \Phi_*(g) \circ \Phi_*(u) \\ &= \Phi(g) \circ \eta_{\Phi_*(A)} \circ \Phi_*(u) = \Phi(g) \circ \Phi(u), \end{aligned}$$

which concludes the proof.  $\square$

**Lemma 6.8.** *Let  $X, Y \in \text{Ob } \mathcal{A}_n$  and let  $A \in \text{Ob } \mathcal{A}$  such that  $\text{lh } A = n$ , let  $f: X \rightarrow Y$  and let  $u: Y \rightarrow A$  in  $\mathcal{M}$ . Then  $\Phi(u \circ f) = \Phi(u) \circ \Phi(f)$ .*

*Proof.* By using (6.7) and the relations on Figure 6.1, we obtain

$$\Phi(u) \circ \Phi(f) = \eta_{\Phi_*(A)} \circ \Phi_*(u) \circ \Phi(f) = \eta_{\Phi_*(A)} \circ \Phi_*(u \circ f) = \Phi(u \circ f),$$

which concludes the proof.  $\square$

At this stage,  $\Phi$ ,  $\varepsilon$ , and  $\mu$  have been extended to the whole category  $\mathcal{A}_{n+1} \cap \mathcal{M}$ . Therefore, arguing by induction on  $n$ , we obtain an extension of  $\Phi$ ,  $\varepsilon$ , and  $\mu$  on  $\mathcal{A} \cap \mathcal{M}$  that satisfies the following:

$$\begin{aligned} \mu_X \circ \varepsilon_X &= \text{id}_X, \quad \Phi(f) \circ \varepsilon_X = \varepsilon_Y \circ f, \quad \text{and } \mu_Y \circ \Phi(f) = f \circ \mu_X, \\ &\text{for every morphism } f: X \rightarrow Y \text{ in } \mathcal{A} \cap \mathcal{M}. \end{aligned} \tag{6.11}$$

**Definition 6.9.** The triple  $\langle \Phi, \varepsilon, \mu \rangle$  thus constructed is the *canonical  $\mathcal{B}$ -cover* of  $\mathcal{A}$ .

The construction  $\langle \Phi, \varepsilon, \mu \rangle$  involves the shelter  $\mathbf{B}$  and categorical operations such as the colimit. Hence, even for fixed  $\mathbf{B}$ , it is defined uniquely only if we choose representatives for colimits of diagrams: otherwise, it is defined only up to isomorphism.

What is still missing is that we do not know yet whether the image under  $\Phi$  of a morphism in  $\mathcal{M}$  is a morphism in  $\mathcal{M}$  (which is why we have, so far, kept this condition out of the induction hypothesis). This is the hardest part of the proof, and it will be the object of the next two sections.

7. FACTORING  $\mathcal{B}$ -LIFTINGS

In this section we shall establish (see Lemma 7.3) a certain ‘‘quasi-universality’’ property of the canonical  $\mathcal{B}$ -cover  $\langle \Phi, \varepsilon, \mu \rangle$  of  $\mathcal{A}$ , with respect to the notion of  $\mathcal{B}$ -lifting introduced in the following definition.

**Definition 7.1.** Let  $A \in \text{Ob } \mathcal{A}$ , let  $\mathcal{I}$  be an ideal of  $\mathcal{A} \upharpoonright A$ , and let  $\Psi: \mathcal{I} \rightarrow \mathcal{B}$  be a functor. A  $\mathcal{B}$ -lifting of  $\Psi$  is a natural transformation from the domain functor  $u \mapsto \text{dom } u$  (from  $\mathcal{I}$  to  $\mathcal{A} \cap \mathcal{M}$ ) to  $\Psi$ .

Hence a  $\mathcal{B}$ -lifting of  $\Psi$  consists of a family  $\tilde{\varepsilon} = \langle \tilde{\varepsilon}_u \mid u \in \mathcal{I} \rangle$ , where  $\tilde{\varepsilon}_u: \text{dom } u \rightarrow \Psi(u)$ , for all  $u \in \mathcal{I}$ , such that if  $u: X \rightarrow A$  and  $v: Y \rightarrow A$  in  $\mathcal{I}$  with  $u \trianglelefteq_A v$ , then, putting  $f = u/v$ , the equality  $\Psi(f) \circ \tilde{\varepsilon}_u = \tilde{\varepsilon}_v \circ f$  holds, see Figure 7.1. Observe that we use the convention, introduced at the beginning of Section 6, to identify  $f$  with  $u \rightarrow v$ , so  $\Psi(f)$  is, in fact, defined as  $\Psi(u \rightarrow v)$ .

$$\begin{array}{ccc} & A & \\ u \nearrow & \swarrow v & \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} \Psi(u) & \xrightarrow{\Psi(f)} & \Psi(v) \\ \tilde{\varepsilon}_u \uparrow & f & \uparrow \tilde{\varepsilon}_v \\ X & \xrightarrow{} & Y \end{array}$$

FIGURE 7.1. Illustrating a  $\mathcal{B}$ -lifting of  $\Psi$ .

**Definition 7.2.** Let  $A \in \text{Ob } \mathcal{A}$ , let  $\mathcal{I}$  be an ideal of  $\mathcal{A} \upharpoonright A$ , and let  $\Psi: \mathcal{I} \rightarrow \mathcal{B}$  be a functor. A factor of  $\langle \Psi, \tilde{\varepsilon} \rangle$  is a natural transformation  $\delta = \langle \delta_u \mid u \in \mathcal{I} \rangle$  from  $\Phi \circ \text{dom}$  to  $\Psi$  such that  $\tilde{\varepsilon}_u = \delta_u \circ \varepsilon_{\text{dom } u}$ , for all  $u \in \mathcal{I}$ .

Hence, for  $u: X \rightarrow A$  and  $v: Y \rightarrow A$  in  $\mathcal{I}$  with  $u \trianglelefteq_A v$  and putting  $f = u/v$ , the diagram of Figure 7.2 commutes.

$$\begin{array}{ccc} \Psi(u) & \xrightarrow{\Psi(f)} & \Psi(v) \\ \uparrow \delta_u & & \uparrow \delta_v \\ \Phi(X) & \xrightarrow{\Phi(f)} & \Phi(Y) \\ \uparrow \varepsilon_X & & \uparrow \varepsilon_Y \\ X & \xrightarrow{f} & Y \end{array}$$

FIGURE 7.2. Illustrating a factor of  $\langle \Psi, \tilde{\varepsilon} \rangle$ .

**Lemma 7.3.** Let  $A \in \text{Ob } \mathcal{A}$  and let  $\mathcal{I}$  and  $\mathcal{J}$  be ideals of  $\mathcal{A} \upharpoonright A$  such that  $\mathcal{J}$  contains  $\mathcal{I}$ . Let  $\tilde{\varepsilon}$  be a  $\mathcal{B}$ -lifting of a functor  $\Psi: \mathcal{J} \rightarrow \mathcal{B}$ . Then any factor of  $\langle \Psi \upharpoonright_{\mathcal{I}}, \tilde{\varepsilon} \upharpoonright_{\mathcal{I}} \rangle$  can be extended to a factor of  $\langle \Psi, \tilde{\varepsilon} \rangle$ .

*Proof.* Arguing by induction on the length reduces the problem to the case where  $\{v \in \mathcal{A} \upharpoonright A \mid v \triangleleft_A u\}$  is contained in  $\mathcal{I}$ , for all  $u \in \mathcal{J}$ . So let  $u: U \rightarrow A$  in  $\mathcal{J} \setminus \mathcal{I}$ , we shall define a morphism  $\delta_u: \Phi(U) \rightarrow \Psi(u)$ .

For all  $v: X \rightarrow U$  and  $w: Y \rightarrow U$  in  $\mathcal{A} \upharpoonright^* U$  such that  $v \trianglelefteq_U w$ , letting  $f = v/w$ , both relations  $u \circ v \triangleleft_A u$  and  $u \circ w \triangleleft_A u$  hold, thus both  $u \circ v$  and  $u \circ w$  belong to  $\mathcal{I}$ . Furthermore, the diagrams of Figure 7.3 commute: this is obvious for the left hand side, while for the right hand side,  $\Psi(v) \circ \delta_{u \circ v} \circ \varepsilon_X = \Psi(v) \circ \tilde{\varepsilon}_{u \circ v} = \tilde{\varepsilon}_u \circ v$ .

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{\delta_{u \circ v}} & \Psi(u \circ v) \xrightarrow{\Psi(v)} \Psi(u) \\ \Phi(f) \downarrow & & \downarrow \Psi(f) \\ \Phi(Y) & \xrightarrow{\delta_{u \circ w}} & \Psi(u \circ w) \xrightarrow{\Psi(w)} \end{array} \quad \begin{array}{ccc} \Phi(X) & \xrightarrow{\Psi(v) \circ \delta_{u \circ v}} & \Psi(u) \\ \uparrow \varepsilon_X & & \uparrow \tilde{\varepsilon}_u \\ X & \xrightarrow{v} & U \end{array}$$

FIGURE 7.3. Putting  $\Psi(u)$  above the diagram defining  $\Phi_*(U)$ .

Hence, by the universal property defining  $\Phi_*(U)$ , there exists a unique morphism  $\gamma_u: \Phi_*(U) \rightarrow \Psi(u)$  such that  $\gamma_u \circ \Phi_*(v) = \Psi(v) \circ \delta_{u \circ v}$ , for all  $v \in \mathcal{A} \upharpoonright^* U$ , and  $\tilde{\varepsilon}_u = \gamma_u \circ \varepsilon^U$ . Put  $\delta_u = \gamma_u^\mathbf{B}$  (see Figure 7.4).

$$\begin{array}{ccc} \Phi_*(U) & \xrightarrow{\gamma_u} & \Psi(u) \\ \uparrow \Phi_*(v) & \uparrow \Psi(v) & \\ \Phi(\text{dom } v) & \xrightarrow[\delta_{u \circ v}]{} & \Psi(u \circ v) \end{array} \quad \begin{array}{ccc} \Phi_*(U) & \xrightarrow{\gamma_u} & \Psi(u) \\ \varepsilon^U \uparrow & \nearrow \tilde{\varepsilon}_u & \\ U & & \end{array} \quad \begin{array}{ccc} \Phi(U) & & \\ \eta_{\Phi_*(U)} \uparrow & \searrow \delta_u = (\gamma_u)^\mathbf{B} & \\ \Phi_*(U) & \xrightarrow[\gamma_u]{} & \Psi(u) \end{array}$$

FIGURE 7.4. Defining  $\gamma_u$  and  $\delta_u$ .

We verify that the  $\delta_u$ -s are as required (see Figure 7.2). First,

$$\delta_u \circ \varepsilon_U = \delta_u \circ \eta_{\Phi_*(U)} \circ \varepsilon^U = \gamma_u \circ \varepsilon^U = \tilde{\varepsilon}_u.$$

Our next series of calculations will prove that the extended  $\delta$  is a natural transformation from  $\Phi \circ \text{dom}$  to  $\Psi$ .

For  $u: U \rightarrow A$  in  $\mathcal{J} \setminus \mathcal{I}$  and  $v: X \rightarrow U$  in  $\mathcal{A} \upharpoonright^* U$  (thus in  $\mathcal{I}$ ),

$$\delta_u \circ \Phi(v) = \delta_u \circ \eta_{\Phi_*(U)} \circ \Phi_*(v) = \gamma_u \circ \Phi_*(v) = \Psi(v) \circ \delta_{u \circ v}.$$

Now let  $u: U \rightarrow A$  and  $v: V \rightarrow A$  in  $\mathcal{J} \setminus \mathcal{I}$  such that  $u \trianglelefteq_A v$ , and put  $f = u/v$ . If  $f$  is not an isomorphism, then  $u \triangleleft_A v$ , thus (since  $v \in \mathcal{J}$ )  $u \in \mathcal{I}$ , a contradiction. Hence  $f$  is an isomorphism. We prove that  $\Psi(f^{-1}) \circ \gamma_v \circ \Phi_*(f)$  satisfies the properties defining  $\gamma_u$ .

$$\begin{aligned} \Psi(f^{-1}) \circ \gamma_v \circ \Phi_*(f) \circ \varepsilon^U &= \Psi(f^{-1}) \circ \gamma_v \circ \varepsilon^V \circ f && \text{(by (6.8) and (6.10))} \\ &= \Psi(f^{-1}) \circ \tilde{\varepsilon}_v \circ f \\ &= \tilde{\varepsilon}_u. \end{aligned}$$

Let  $w \in \mathcal{A} \upharpoonright^* U$ . By using (6.8) and (6.10), we compute:

$$\begin{aligned} \Psi(f^{-1}) \circ \gamma_v \circ \Phi_*(f) \circ \Phi_*(w) &= \Psi(f^{-1}) \circ \gamma_v \circ \Phi_*(f \circ w) \\ &= \Psi(f^{-1}) \circ \Psi(f \circ w) \circ \delta_{u \circ w} \\ &= \Psi(w) \circ \delta_{u \circ w}. \end{aligned}$$

Therefore,  $\Psi(f^{-1}) \circ \gamma_v \circ \Phi_*(f) = \gamma_u$ , that is,  $\gamma_v \circ \Phi_*(f) = \Psi(f) \circ \gamma_u$ . Now we can compute further, using the assumption that  $\mathbf{B}$  is a shelter:

$$\begin{aligned}\delta_u &= \gamma_u^{\mathbf{B}} = (\Psi(f^{-1}) \circ \gamma_v \circ \Phi_*(f))^{\mathbf{B}} \\ &= \Psi(f^{-1}) \circ \gamma_v^{\mathbf{B}} \circ \mathbf{B}(\Phi_*(f)) \\ &= \Psi(f^{-1}) \circ \delta_v \circ \Phi(f),\end{aligned}$$

whence  $\Psi(f) \circ \delta_u = \delta_v \circ \Phi(f)$ .  $\square$

## 8. PRESERVATION OF $\mathcal{M}$ BY $\Phi$

In this section we shall prove the remaining claim about the canonical  $\mathcal{B}$ -cover  $\langle \Phi, \varepsilon, \mu \rangle$ , namely, that  $\Phi$  preserves  $\mathcal{M}$ . The idea of the proof is the following. For  $f: U \rightarrow A$  in  $\mathcal{A} \cap \mathcal{M}$ , we construct, using the amalgamation property stated in Lemma 5.4, a certain functor  $\Psi$ , defined on all subobjects of  $A$ , together with a  $\mathcal{B}$ -lifting  $\tilde{\varepsilon}$  of  $\Psi$ . Furthermore, we shall see that the restriction of  $\langle \Psi, \tilde{\varepsilon} \rangle$  to all subobjects of  $A$  below  $f$  has a factor. By the “quasi-universality” property established in Section 7, namely, Lemma 7.3, this factor extends to a factor of  $\langle \Psi, \tilde{\varepsilon} \rangle$  on all subobjects of  $A$ . As  $\Psi$  is constructed in such a way that the arrow  $\Psi(f \rightarrow \text{id}_A)$  belongs to  $\mathcal{M}$ , it follows that  $\Phi(f)$  also belongs to  $\mathcal{M}$ .

**Lemma 8.1.** *For any morphism  $f$  of  $\mathcal{A} \cap \mathcal{M}$ ,  $\Phi(f)$  belongs to  $\mathcal{B} \cap \mathcal{M}$ .*

*Proof.* We let  $f: U \rightarrow A$  in  $\mathcal{A} \cap \mathcal{M}$ , we prove that  $\Phi(f) \in \mathcal{M}$ . If  $f$  is an isomorphism, then (since  $\Phi$  is a functor) so is  $\Phi(f)$ , thus  $\Phi(f) \in \mathcal{M}$ . From now on we assume that  $f$  is not an isomorphism.

Since  $\varepsilon_U: U \rightarrow \Phi(U)$  is a section (for  $\mu_U \circ \varepsilon_U = \text{id}_U$ ), it follows from Lemma 5.4 that there exists  $B \in \text{Ob } \mathcal{B}$ , together with  $\varphi: \Phi(U) \rightarrow B$  and  $\varepsilon: A \rightarrow B$  in  $\mathcal{M}$ , such that  $\varphi \circ \varepsilon_U = \varepsilon \circ f$ .

For each  $u: X \rightarrow A$  in  $\mathcal{A} \upharpoonright A$ , we define  $\Psi(u) \in \text{Ob } \mathcal{B}$  and  $\tilde{\varepsilon}_u: X \rightarrow \Psi(u)$  by

$$\Psi(u) = \begin{cases} \Phi(X), & \text{if } u \leq_A f, \\ B, & \text{otherwise;} \end{cases} \quad \tilde{\varepsilon}_u = \begin{cases} \varepsilon_X, & \text{if } u \leq_A f, \\ \varepsilon \circ u, & \text{otherwise.} \end{cases} \quad (8.1)$$

For  $u: X \rightarrow A$  and  $v: Y \rightarrow A$  in  $\mathcal{A} \upharpoonright A$ , we put  $g = u/v$  and we define a morphism  $\Psi(u \rightarrow v): \Psi(u) \rightarrow \Psi(v)$  in  $\mathcal{B}$  as follows:

**Case 1.**  $v \leq_A f$ . Put  $\Psi(u \rightarrow v) = \Phi(g)$ .

**Case 2.**  $u \leq_A f$ ,  $v \not\leq_A f$ . Put  $\Psi(u \rightarrow v) = \varphi \circ \Phi(u/f)$ .

**Case 3.**  $u, v \not\leq_A f$ . Put  $\Psi(u \rightarrow v) = \text{id}_B$ .

**Claim 1.** *In the context above,  $\Psi(u \rightarrow v) \circ \tilde{\varepsilon}_u = \tilde{\varepsilon}_v \circ g$ .*

*Proof of Claim.* In Case 1, this is equivalent to the statement  $\Phi(g) \circ \varepsilon_X = \varepsilon_Y \circ g$ , which holds.

In Case 2, putting  $\overline{u} = u/f$ , we compute

$$\varphi \circ \Phi(\overline{u}) \circ \varepsilon_X = \varphi \circ \varepsilon_U \circ \overline{u} = \varepsilon \circ f \circ \overline{u} = \varepsilon \circ u = \varepsilon \circ v \circ g,$$

which is the desired statement.

In Case 3, from  $v \circ g = u$  it follows that  $\varepsilon \circ v \circ g = \varepsilon \circ u$ , which is the desired statement.  $\square$  Claim 1.

**Claim 2.**  *$\Psi$  is a functor.*

*Proof of Claim.* It suffices to prove that  $\Psi(u \rightarrow w) = \Psi(v \rightarrow w) \circ \Psi(u \rightarrow v)$ , for all  $u \trianglelefteq_A v \trianglelefteq_A w$  in  $\mathcal{A} \upharpoonright A$ . Put  $X = \text{dom } u$ ,  $Y = \text{dom } v$ ,  $Z = \text{dom } w$ ,  $g = u/v$ , and  $h = v/w$ . We separate cases.

**Case 1.**  $w \trianglelefteq_A f$ . Then

$$\Psi(v \rightarrow w) \circ \Psi(u \rightarrow v) = \Phi(h) \circ \Phi(g) = \Phi(h \circ g) = \Psi(u \rightarrow w).$$

**Case 2.**  $v \trianglelefteq_A f$  and  $w \not\trianglelefteq_A f$ . Put  $\bar{u} = u/f$  and  $\bar{v} = v/f$ . The equality  $u = v \circ g$  can be written  $f \circ \bar{u} = f \circ \bar{v} \circ g$ , thus, since  $f$  is monic,  $\bar{u} = \bar{v} \circ g$ . Therefore,

$$\Psi(u \rightarrow w) = \varphi \circ \Phi(\bar{u}) = \varphi \circ \Phi(\bar{v}) \circ \Phi(g) = \Psi(v \rightarrow w) \circ \Psi(u \rightarrow v).$$

**Case 3.**  $u \trianglelefteq_A f$  and  $v \not\trianglelefteq_A f$ . Put  $\bar{u} = u/f$ . Then

$$\Psi(u \rightarrow w) = \varphi \circ \Phi(\bar{u}) = \text{id}_B \circ \varphi \circ \Phi(\bar{u}) = \Psi(v \rightarrow w) \circ \Psi(u \rightarrow v).$$

**Case 4.**  $u \not\trianglelefteq_A f$ . Then  $\Psi(u \rightarrow v) = \Psi(u \rightarrow w) = \Psi(v \rightarrow w) = \text{id}_B$ , whence

$$\Psi(u \rightarrow w) = \Psi(v \rightarrow w) \circ \Psi(u \rightarrow v).$$

This concludes the proof of our claim.  $\square$  Claim 2.

By Claims 1 and 2,  $\tilde{\varepsilon}$  is a  $\mathcal{B}$ -lifting of the functor  $\Psi$  on the ideal  $\mathcal{J} = \mathcal{A} \upharpoonright A$ . Furthermore, putting

$$\mathcal{I} = \{u \in \mathcal{A} \upharpoonright A \mid u \trianglelefteq_A f\},$$

the rule  $\delta_u = \text{id}_{\Phi(\text{dom } u)}$  defines a factor of  $\langle \Psi \upharpoonright_{\mathcal{I}}, \tilde{\varepsilon}_{\mathcal{I}} \rangle$ . Therefore, by Lemma 7.3,  $\delta$  extends to a factor of  $\langle \Psi, \tilde{\varepsilon} \rangle$ , which we shall still denote by  $\delta$ .

Since  $f \in \mathcal{I}$  and  $\text{id}_A \notin \mathcal{I}$ , we obtain  $\Psi(f \rightarrow \text{id}_A) = \varphi \circ \Phi(\text{id}_U) = \varphi$ . Therefore,

$$\varphi = \varphi \circ \delta_f = \delta_{\text{id}_A} \circ \Phi(f).$$

Since  $\varphi \in \mathcal{M}$ , we obtain that  $\Phi(f) \in \mathcal{M}$ .  $\square$

This completes the proof of Theorem 5.3: the canonical  $\mathcal{B}$ -cover  $\langle \Phi, \varepsilon, \mu \rangle$  is a solution of the given problem.

## 9. DISTRIBUTIVE AND ULTRABOOLEAN SEMILATTICES

A  $\langle \vee, 0 \rangle$ -semilattice  $\langle S, \vee, 0, \leq \rangle$  is *distributive*, if its ideal lattice  $\text{Id } S$  is distributive, see G. Grätzer [6, Section II.5]. Equivalently, for all  $a, b, c \in S$ , if  $c \leq a \vee b$ , then there are  $x \leq a$  and  $y \leq b$  such that  $c = x \vee y$ . Distributive  $\langle \vee, 0 \rangle$ -semilattices are characterized in P. Pudlák [12, Fact 4, p. 100] as directed  $\langle \vee, 0 \rangle$ -unions of finite distributive  $\langle \vee, 0 \rangle$ -semilattices.

Denote by  $\mathcal{S}$  the category of finite  $\langle \vee, 0 \rangle$ -semilattices and  $\langle \vee, 0 \rangle$ -homomorphisms, and by  $\mathcal{D}$  and  $\mathcal{B}$  the full subcategories of  $\mathcal{S}$  consisting of all distributive, respectively Boolean members of  $\mathcal{S}$ . We denote by  $\mathcal{M}$  the subcategory of  $\mathcal{S}$  consisting of all  $\langle \vee, 0 \rangle$ -embeddings. Of course,  $\mathcal{M}$  is an ideal of monics of  $\mathcal{S}$ .

**Lemma 9.1.** *Let  $S, T$ , and  $D$  be  $\langle \vee, 0 \rangle$ -semilattices with  $D$  finite distributive, and let  $e: S \rightarrow T$  and  $f: S \rightarrow D$  be  $\langle \vee, 0 \rangle$ -homomorphisms with  $e$  an embedding. Then there exists a largest  $\langle \vee, 0 \rangle$ -homomorphism  $g: T \rightarrow D$  extending  $f$ , and it is given by the formula*

$$g(t) = \bigwedge^D (f(s) \mid s \in S, t \leq e(s)), \text{ for all } t \in T. \quad (9.1)$$

(Of course, the meet of the empty set is defined here as the unit of  $D$ .)

*Proof.* An easy exercise. Although the distributivity of  $D$  is not used for correctness of the definition (9.1), it is used for proving that  $g$  is a  $\langle \vee, 0 \rangle$ -homomorphism.  $\square$

We shall call the map  $g$  defined in (9.1) the *largest extension* of  $f$  with respect to  $e$ .

Any  $S \in \text{Ob } \mathcal{S}$  is a finite lattice, thus also a meet-semilattice. We put  $\mathbf{B}(S) = \mathfrak{P}(\text{M}(S))$ , and we let

$$\eta_S: S \hookrightarrow \mathbf{B}(S), \quad a \mapsto \{u \in \text{M}(S) \mid a \not\leq u\}.$$

**Lemma 9.2** (folklore). *The map  $\eta_S$  is a  $\langle \vee, 0, 1 \rangle$ -embedding from  $S$  into  $\langle \mathbf{B}(S), \cup, \emptyset, S \rangle$ , for every finite  $\langle \vee, 0 \rangle$ -semilattice  $S$ .*

For an isomorphism  $f: S \rightarrow T$  of finite  $\langle \vee, 0 \rangle$ -semilattices, we put

$$\mathbf{B}(f)(X) = f[X], \quad \text{for all } X \in \mathbf{B}(S).$$

It is immediate to verify that  $\mathbf{B}$  is a functor from  $\mathcal{S}^{\text{iso}}$  to  $\mathcal{B}^{\text{iso}}$  and that  $\eta$  is a natural transformation from the identity of  $\mathcal{S}^{\text{iso}}$  to  $\mathbf{B}$ .

**Definition 9.3.** Let  $S$  and  $A$  be finite  $\langle \vee, 0 \rangle$ -semilattices with  $A$  distributive, and let  $g: S \rightarrow A$  be a  $\langle \vee, 0 \rangle$ -homomorphism. We denote  $g^{\mathbf{B}}$  the largest  $\langle \vee, 0 \rangle$ -extension of  $g$  from  $\mathbf{B}(S)$  to  $A$  with respect to the embedding  $\eta_S: S \hookrightarrow \mathbf{B}(S)$ .

Now the proof of the following lemma is a straightforward exercise (see Definition 4.2). Items (i), (ii), (1), and (2) of Definition 4.2 follows from the fact that the formulas defining  $\mathbf{B}$ ,  $g \mapsto g^{\mathbf{B}}$ , and  $\eta$  are ‘intrinsic’, thus preserved under isomorphisms. Item (iii) of Definition 4.2 follows from Lemma 9.1 (see Definition 9.3).

**Lemma 9.4.** *The correspondences  $\mathbf{B}$ ,  $\eta$  described above define a shelter of  $\mathcal{S}$  by  $\mathcal{B}$  with respect to  $\langle \mathcal{D}, \mathcal{M} \rangle$ .*

The corresponding commutative diagram is given on the right hand side of Figure 4.1. Now we are ready to prove our main semilattice-theoretical result.

**Theorem 9.5.** *There exists a functorial retraction  $\langle \Phi, \varepsilon, \mu \rangle$  of the category  $\mathcal{D} \cap \mathcal{M}$  to the category  $\mathcal{B} \cap \mathcal{M}$ . Furthermore,  $\varepsilon_A$  is a  $\langle \vee, 0, 1 \rangle$ -embedding, for all  $A \in \text{Ob } \mathcal{D}$ .*

*Proof.* We prove that the assumptions of Theorem 5.3 are satisfied, where we replace  $\mathcal{C}$  by  $\mathcal{S}$  and  $\mathcal{A}$  by  $\mathcal{D}$ . Item (i) is a very particular case of the well-known fact that every variety of algebras has small colimits, see, for example, [2, Theorem 8.3.8]. Item (ii) is trivial. Item (iii) is Lemma 9.4.

Finally, it is proved in [8, Theorem 2.10] that every semilattice embeds into an injective semilattice. Hence the variety of semilattices has the so-called Transfer Property (see [9, Proposition 1.5]), thus *a fortiori* the Amalgamation Property. Since every finitely generated semilattice is finite, these results extend to the finite case. Technically speaking, these results are established in [8, 9] for semilattices which do not necessarily have a unit; however, the extension to the case with unit is trivial. The result for  $\langle \vee, 0 \rangle$ -semilattices is dual. This obviously implies Assumption (iv) of Theorem 5.3.

It remains to establish that the maps  $\varepsilon_A$  constructed in the proof of Theorem 5.3 are 1-preserving. We argue by induction on  $\text{lh } A$ . By definition,  $\Phi_*(A)$  is  $\langle \vee, 0 \rangle$ -generated by the set

$$G = \text{im } \varepsilon^A \cup \bigcup (\text{im } \Phi_*(u) \mid u \in \mathcal{A} \upharpoonright^* A). \quad (9.2)$$

Let  $u: X \hookrightarrow A$  in  $\mathcal{A} \upharpoonright^* A$ . By the induction hypothesis,  $\varepsilon_X$  is 1-preserving, thus

$$\Phi_*(u)(1_{\Phi(X)}) = \Phi_*(u) \circ \varepsilon_X(1_X) = \varepsilon^A \circ u(1_X) \leq \varepsilon^A(1_A),$$

thus the largest element of  $X$  is  $\varepsilon^A(1_A)$ . Hence the largest element of  $\Phi_*(A)$  is also  $\varepsilon^A(1_A)$ , that is,  $\varepsilon^A$  is 1-preserving. Since  $\eta_S$  is 1-preserving for all  $S$ , it follows that  $\varepsilon_A = \eta_{\Phi_*(A)} \circ \varepsilon^A$  is also 1-preserving.  $\square$

Finally, denote by  $b_A$  the largest  $b \in \Phi(A)$  such that  $\mu_A(b) = 0$ , for every finite distributive  $\langle \vee, 0 \rangle$ -semilattice  $A$ . After replacing  $\Phi(A)$  by its interval  $[b_A, 1]$ , and this for all  $A$ , we obtain that the  $\mu_A$ -s may be assumed to separate zero (i.e.,  $\mu_A^{-1}\{0\} = \{0\}$ ).

For convenience, we list here the properties satisfied by the triple  $\langle \Phi, \varepsilon, \mu \rangle$  of Theorem 9.5:

- The correspondence  $\Phi$  is a functor from the category  $\mathcal{D} \cap \mathcal{M}$  of all finite distributive  $\langle \vee, 0 \rangle$ -semilattices with  $\langle \vee, 0 \rangle$ -embeddings to the category  $\mathcal{B} \cap \mathcal{M}$  of all finite Boolean  $\langle \vee, 0 \rangle$ -semilattices with  $\langle \vee, 0 \rangle$ -embeddings.
- The map  $\varepsilon_A$  is a  $\langle \vee, 0, 1 \rangle$ -embedding from  $A$  into  $\Phi(A)$  and the map  $\mu_A$  is a zero-separating  $\langle \vee, 0, 1 \rangle$ -homomorphism from  $\Phi(A)$  onto  $A$ , for every finite distributive  $\langle \vee, 0 \rangle$ -semilattice  $A$ . Furthermore,  $\mu_A \circ \varepsilon_A = \text{id}_A$ .
- For every  $\langle \vee, 0 \rangle$ -embedding  $f: X \hookrightarrow Y$  between finite distributive  $\langle \vee, 0 \rangle$ -semilattices  $X$  and  $Y$ , both equalities  $\Phi(f) \circ \varepsilon_X = \varepsilon_Y \circ f$  and  $\mu_Y \circ \Phi(f) = f \circ \mu_X$  hold.

Observe that these properties imply that  $\Phi(f)$  preserves the unit whenever  $f$  does.

As shown by the following result, this functorial inverse of the functor  $\Pi$  does not arise from an adjunction.

**Proposition 9.6.** *The projection functor  $\Pi: \text{Retr}(\mathcal{D} \cap \mathcal{M}, \mathcal{B} \cap \mathcal{M}) \rightarrow \mathcal{D} \cap \mathcal{M}$  has neither a right nor a left adjoint.*

*Proof.* Denote the objects of  $\mathcal{R} = \text{Retr}(\mathcal{D} \cap \mathcal{M}, \mathcal{B} \cap \mathcal{M})$  as  $p = \langle D_p, B_p, \alpha_p, \beta_p \rangle$ . A left or right adjoint of  $\Pi$  is given by a functor  $\Psi: \mathcal{D} \rightarrow \mathcal{R}$ . Let  $\Psi$  be given, for any  $f: D \rightarrow E$  in  $\mathcal{D} \cap \mathcal{M}$ , by

$$\begin{aligned} \Psi(D) &= \left\langle \tilde{D}, \Phi(D), \varepsilon^D, \mu^D \right\rangle \text{ and } \Psi(f) = \left\langle \tilde{f}, \Phi(f) \right\rangle, \\ &\text{where } \tilde{f}: \tilde{D} \rightarrow \tilde{E} \text{ and } \Phi(f): \Phi(D) \rightarrow \Phi(E). \end{aligned} \tag{9.3}$$

Suppose first that  $\Psi$  is a left adjoint of  $\Pi$ , and denote by  $\eta$  the unit of the corresponding adjunction. So  $\eta_D: D \hookrightarrow \tilde{D}$ , for every  $D \in \text{Ob } \mathcal{D}$ . By the definition of an adjunction, for all  $D \in \text{Ob } \mathcal{D}$ ,  $p \in \text{Ob } \mathcal{R}$ , and  $f: D \hookrightarrow D_p$ , there exists a unique  $\langle f_*, f^* \rangle: \Psi(D) \rightarrow p$  such that  $f = f_* \circ \eta_D$ . In particular, for  $D = D_p$  and  $f = \text{id}_D$  (as seen in the Introduction, there are always  $B$ ,  $\alpha$ , and  $\beta$  such that  $\langle D, B, \alpha, \beta \rangle \in \text{Ob } \mathcal{R}$ ), we obtain, since all our semilattices are finite, that  $\eta_D$  is an isomorphism from  $D$  onto  $\tilde{D}$ .

Now let  $D = \mathbf{3}$  (the three-element chain),  $D_p = B_p = \mathbf{2}^2$ ,  $\alpha_p = \beta_p = \text{id}_{\mathbf{2}^2}$ , and  $f: \mathbf{3} \hookrightarrow \mathbf{2}^2$  any  $\langle \vee, 0 \rangle$ -embedding. From  $\langle f_*, f^* \rangle: \Psi(D) \rightarrow p$  it follows that  $f_* \circ \mu^D = f^*$ , an embedding. Thus  $\mu^D$  is an embedding, and so  $\Phi(D) \cong \tilde{D} \cong D \cong \mathbf{3}$ , which is not Boolean; a contradiction.

Now suppose that  $\Psi$  is a right adjoint of  $\Pi$ , and denote by  $\varepsilon$  the counit of the corresponding adjunction. So  $\varepsilon_D: \tilde{D} \hookrightarrow D$ , for all  $D \in \text{Ob } \mathcal{D}$ . By the definition of an adjunction, for all  $D \in \text{Ob } \mathcal{D}$ ,  $p \in \text{Ob } \mathcal{R}$ , and  $f: D_p \hookrightarrow D$ , there exists a unique

$\langle f_*, f^* \rangle : p \rightarrow \Psi(D)$  such that  $\varepsilon_D \circ f_* = f$ . In particular,  $f^* : B_p \hookrightarrow \Phi(D)$ , and hence  $|B_p| \leq |\Phi(D)|$ . However, for  $D$  and  $D_p$  fixed,  $|B_p|$  can be taken arbitrarily large, a contradiction.  $\square$

The functorial retraction given by Theorem 9.5 is given by an explicit formula. This makes it possible to give a crude upper bound for the maximum  $\varphi(n)$  of all cardinalities of  $\Phi(A)$ , where  $A$  is a distributive  $\langle \vee, 0 \rangle$ -semilattice of cardinality at most  $n$ . Of course,  $\varphi(1) = 1$ . As (9.2) gives a generating subset of  $\Phi_*(A)$ , we obtain

$$|\Phi_*(A)| \leq 2^{n+1+2^n\varphi(n)},$$

whence

$$\varphi(n+1) \leq 2^{2^{n+1+2^n\varphi(n)}}.$$

Hence  $\varphi(n)$  is, roughly speaking, majorized by a tower of exponentials of length  $2n$ , which is, of course, beyond the reach of any implementation.

As illustrated in [15], the poset of distributive subsemilattices of a finite distributive  $\langle \vee, 0 \rangle$ -semilattice can be quite complicated. To the contrary, the corresponding structure is much nicer for Boolean subsemilattices. This motivates the definition of ultraboolean introduced in Section 1.

**Corollary 9.7.** *Every distributive  $\langle \vee, 0 \rangle$ -semilattice  $D$  is a  $\langle \vee, 0 \rangle$ -retract of an ultraboolean  $\langle \vee, 0 \rangle$ -semilattice  $B$ . Furthermore, if  $D$  has a unit, then  $B$  can be taken with a unit.*

*Proof.* By Pudlák's Lemma,  $D = \varinjlim_{i \in I} D_i$ , for a direct system  $\langle D_i, f_{i,j} \mid i \leq j \text{ in } I \rangle$  of finite distributive  $\langle \vee, 0 \rangle$ -semilattices and  $\langle \vee, 0 \rangle$ -embeddings  $f_{i,j} : D_i \rightarrow D_j$ . Furthermore, we may assume that all  $D_i$ -s contain as an element the unit of  $D$  in case there is any.

Now we use the functorial retraction constructed in the proof of Theorem 9.5. The semilattice  $B = \varinjlim_{i \in I} \Phi(D_i)$ , with transition maps  $\Phi(f_{i,j}) : \Phi(D_i) \rightarrow \Phi(D_j)$ , is ultraboolean and has a unit in case  $D$  has a unit. Furthermore, the natural transformations  $\langle \varepsilon_{D_i} \mid i \in I \rangle$  and  $\langle \mu_{D_i} \mid i \in I \rangle$  define, by direct limit,  $\langle \vee, 0 \rangle$ -homomorphisms  $\varepsilon : D \rightarrow B$  and  $\mu : B \rightarrow D$  such that  $\mu \circ \varepsilon = \text{id}_D$ . Therefore,  $D$  is a retract of  $B$ . If  $D$  has a unit, then  $\varepsilon(1_D) = 1_B$ .  $\square$

## 10. SIMULTANEOUS LATTICE EMBEDDINGS INTO FINITE BOOLEAN SEMILATTICES

The maps  $\varepsilon_X : X \rightarrow \Phi(X)$  constructed in the proof of Theorem 9.5 are  $\langle \vee, 0, 1 \rangle$ -embeddings. On the other hand, for every finite distributive  $\langle \vee, 0 \rangle$ -semilattice  $S$ , the embedding from  $S$  into  $\mathfrak{P}(J(S))$  that with every  $a \in S$  associates the set  $\{p \in J(S) \mid p \leq a\}$  is always a lattice embedding, and it has nice ‘‘almost functorial’’ properties, see [13, Section 1]. Hence the question whether a new functorial retraction may be constructed, with the corresponding maps  $\varepsilon_X$ -s being *lattice* homomorphisms, is natural.

In the present section we shall prove, by a counterexample, that this is not possible. In fact we shall prove a much stronger negative statement, see Example 10.3.

All direct systems considered in this section will be indexed by posets. Hence, if  $\langle I, \leq \rangle$  is a poset, an  $I$ -indexed direct system in a category  $\mathcal{A}$  consists of a system  $\langle A_i, f_{i,j} \mid i \leq j \text{ in } I \rangle$ , with  $f_{i,j} : A_i \rightarrow A_j$  in  $\mathcal{A}$ , for  $i \leq j$  in  $I$ , such that  $f_{i,i} = \text{id}_{A_i}$  and  $f_{i,k} = f_{j,k} \circ f_{i,j}$ , for all  $i \leq j \leq k$  in  $I$ .

Suppose now that all  $A_i$ -s are finite  $\langle \vee, 0 \rangle$ -semilattices, all the  $f_{i,j}$ -s are  $\langle \vee, 0 \rangle$ -embeddings, and let  $i \leq j$  in  $I$ . For any  $q \in J(A_j)$ , we define  $\partial^{i,j}q$  as the set of all minimal  $p \in A_i$  such that  $q \leq f_{i,j}(p)$ . Of course,  $\partial^{i,j}q$  is a subset of  $J(A_i)$ .

**Definition 10.1.** Let  $\alpha = \langle A_i, f_{i,j} \mid i \leq j \text{ in } I \rangle$  and  $\beta = \langle B_i, g_{i,j} \mid i \leq j \text{ in } I \rangle$  be direct systems of lattices, indexed by the same poset  $I$ . A *simultaneous lattice embedding* from  $\alpha$  into  $\beta$  is a system  $\langle \varepsilon_i \mid i \in I \rangle$  of lattice embeddings  $\varepsilon_i: A_i \hookrightarrow B_i$  such that  $\varepsilon_j \circ f_{i,j} = g_{i,j} \circ \varepsilon_i$ , for all  $i \leq j$  in  $I$ .

**Proposition 10.2.** Let  $\langle A_i, f_{i,j} \mid i \leq j \text{ in } I \rangle$  be a direct system, indexed by a poset  $I$ , of finite distributive  $\langle \vee, 0 \rangle$ -semilattices and  $\langle \vee, 0 \rangle$ -embeddings, that admits a simultaneous lattice embedding into a direct system of finite Boolean semilattices. Then for all  $i \leq j$  in  $I$  and all  $p \in J(A_i)$ , there exists  $q \leq f_{i,j}(p)$  in  $J(A_j)$  such that the following statements hold:

- (i)  $\partial^{i,j}q = \{p\}$ .
- (ii) For all  $k \in I$  with  $i \leq k \leq j$  and all  $r \in \partial^{k,j}q$ , the following implication holds:

$$r \leq f_{i,k}(1_{A_i}) \implies r \leq f_{i,k}(p).$$

*Proof.* We fix a simultaneous lattice embedding as in Definition 10.1, with all the  $A_i$ -s finite distributive  $\langle \vee, 0 \rangle$ -semilattices and all the  $B_i$ -s finite Boolean.

We put  $0_i = 0_{A_i}$  and  $1_i = 1_{A_i}$ , for all  $i \in I$ . Furthermore, define  $X_i$  as the set of all atoms of  $B_i$ . Replacing  $B_i$  by the interval  $[\varepsilon_i(0_i), \varepsilon_i(1_i)]$  (which is still Boolean), we see that there is no loss of generality in assuming that  $\varepsilon_i$  is a  $\langle \vee, \wedge, 0, 1 \rangle$ -embedding, for all  $i \in I$ . Furthermore, we may assume that the  $X_i$ -s are pairwise disjoint and that all the  $f_{i,j}$ -s are set-theoretical inclusion mappings, so that  $A_i \subseteq A_j$ , for all  $i \leq j$  in  $I$ .

Since  $\varepsilon_i$  is a  $(\wedge, 1)$ -homomorphism, we can define  $\mu_i(x)$ , for  $x \in B_i$ , as the least  $a \in A_i$  such that  $x \leq \varepsilon_i(a)$ . Hence,

$$x \leq \varepsilon_i(a) \iff \mu_i(x) \leq a, \text{ for all } \langle a, x \rangle \in A_i \times B_i. \quad (10.1)$$

For  $\langle \xi, \eta \rangle \in X_i \times X_j$ , let  $\xi \trianglelefteq \eta$  hold, if  $\eta \leq g_{i,j}(\xi)$ . It is obvious that  $\trianglelefteq$  is a partial ordering on  $\bigcup(X_i \mid i \in I)$ .

The following claim records a few elementary facts.

**Claim 1.**

- (i)  $\mu_i \circ \varepsilon_i = \text{id}_{A_i}$ .
- (ii)  $x \leq \varepsilon_i \circ \mu_i(x)$ , for all  $x \in B_i$ .
- (iii)  $\mu_i(\xi) \in J(A_i)$ , for all  $\xi \in X_i$ .
- (iv)  $\varepsilon_i(a) = \bigvee (\xi \in X_i \mid \mu_i(\xi) \leq a)$ , for all  $a \in A_i$ .
- (v)  $\mu_j \circ g_{i,j}(x) \leq \mu_i(x)$ , for all  $x \in B_i$ .
- (vi)  $\xi \trianglelefteq \eta$  implies that  $\mu_j(\eta) \leq \mu_i(\xi)$ , for all  $\langle \xi, \eta \rangle \in X_i \times X_j$ .

*Proof.* It follows from (10.1) that  $\varepsilon_i(a) \leq \varepsilon_i(b)$  iff  $\mu_i \varepsilon_i(a) \leq b$ , for all  $a, b \in A_i$ . Since  $\varepsilon_i$  is an embedding, (i) follows.

Substituting  $a = \mu_i(x)$  in (10.1) gives immediately (ii).

Put  $p = \mu_i(\xi)$ , for  $\xi \in X_i$ . Since  $\xi$  is nonzero, so is  $p$ . Let  $p = a \vee b$ , where  $a, b \in A_i$ . So,  $\xi \leq \varepsilon_i(p) = \varepsilon_i(a) \vee \varepsilon_i(b)$ , but  $\xi$  is an atom of  $B_i$ , whence either  $\xi \leq \varepsilon_i(a)$  or  $\xi \leq \varepsilon_i(b)$ , and hence, by the definition of  $p$ , either  $p = a$  or  $p = b$ . Item (iii) follows.

By (10.1), an element  $\xi$  of  $X_i$  lies below  $\varepsilon_i(a)$  iff  $\mu_i(\xi) \leq a$ . Since  $B_i$  is Boolean, (iv) follows.

From (ii) it follows that  $g_{i,j}(x) \leq g_{i,j} \circ \varepsilon_i \circ \mu_i(x) = \varepsilon_j \circ \mu_i(x)$ ; item (v) follows.

For  $\langle \xi, \eta \rangle \in X_i \times X_j$  with  $\xi \sqsubseteq \eta$ , that is,  $\eta \leq g_{i,j}(\xi)$ , we obtain, using (v), that  $\mu_j(\eta) \leq \mu_j \circ g_{i,j}(\xi) \leq \mu_i(\xi)$ . Item (vi) follows.  $\square$

**Claim 2.** *The set  $\partial^{i,j} \mu_j(\eta)$  is contained in  $\{\mu_i(\xi) \mid \xi \in X_i \text{ and } \xi \sqsubseteq \eta\}$ , for all  $i \leq j$  in  $I$  and all  $\eta \in X_j$ .*

*Proof of Claim.* Let  $p \in \partial^{i,j} \mu_j(\eta)$ . From Claim 1(iv) it follows that

$$\varepsilon_j(p) = \bigvee (\beta \in X_j \mid \mu_j(\beta) \leq p),$$

whence  $\eta \leq \varepsilon_j(p)$ . Therefore, again by using Claim 1(iv), we obtain

$$\eta \leq g_{i,j} \circ \varepsilon_i(p) = g_{i,j} \left( \bigvee (\xi \in X_i \mid \mu_i(\xi) \leq p) \right) = \bigvee (g_{i,j}(\xi) \mid \xi \in X_i \text{ and } \mu_i(\xi) \leq p),$$

whence there exists  $\xi \in X_i$  such that  $\xi \sqsubseteq \eta$  and  $\mu_i(\xi) \leq p$ . Hence, by Claim 1(vi),  $\mu_j(\eta) \leq \mu_i(\xi) \leq p$ , with  $\mu_i(\xi) \in A_i$  and  $p \in \partial^{i,j} \mu_j(\eta)$ . Therefore,  $p = \mu_i(\xi)$ .  $\square$  Claim 2.

Now we can conclude the proof of Proposition 10.2. It follows from Claim 1(i,iv) that

$$p = \mu_i \circ \varepsilon_i(p) = \bigvee (\mu_i(\xi) \mid \xi \in X_i \text{ and } \mu_i(\xi) \leq p),$$

thus, since  $p$  is join-irreducible, there exists  $\xi \in X_i$  such that  $p = \mu_i(\xi)$ . Since  $g_{i,j}$  is an embedding, there exists  $\eta \in X_j$  such that  $\eta \leq g_{i,j}(\xi)$  and  $\eta \not\leq g_{i,j}(\neg^{B_i} \xi)$ . This means that  $\xi \sqsubseteq \eta$  and  $\xi' \not\sqsubseteq \eta$  for all  $\xi' \in X_i \setminus \{\xi\}$ . We prove that the element  $q = \mu_j(\eta)$  is as desired.

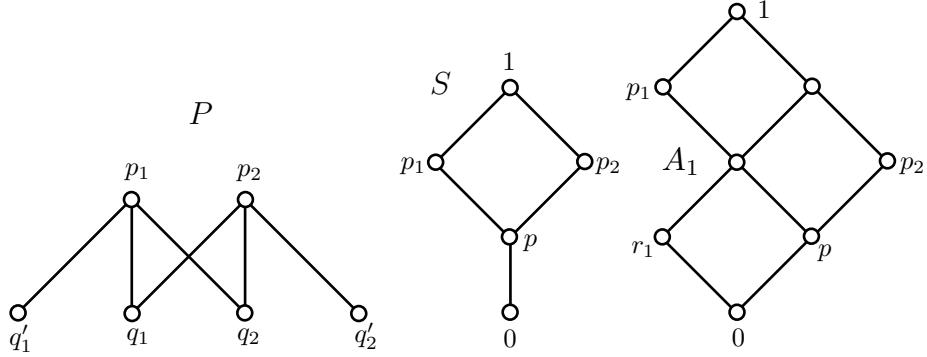
First, by Claim 1(iii),  $q$  belongs to  $J(A_j)$ . Since  $q \leq p$ , the set  $\partial^{i,j} q$  is nonempty. Let  $p' \in \partial^{i,j} q$ . By Claim 2, there exists  $\xi' \in X_i$  such that  $\xi' \sqsubseteq \eta$  and  $\mu_i(\xi') = p'$ . By the definition of  $\eta$ , we obtain that  $\xi' = \xi$ , so  $p' = p$ . Hence,  $\partial^{i,j} q = \{p\}$ .

Now let  $k \in I$  with  $i \leq k \leq j$  and let  $r \in \partial^{k,j} q$  with  $r \leq 1_i$ . The latter inequality implies that  $\partial^{i,k} r$  is nonempty. Let  $p' \in \partial^{i,k} r$ . Since  $r \in \partial^{k,j} q = \partial^{k,j} \mu_j(\eta)$ , there exists, by Claim 2,  $\zeta \in X_k$  such that  $r = \mu_k(\zeta)$  and  $\zeta \sqsubseteq \eta$ . Since  $p' \in \partial^{i,k} r$ , there exists, again by Claim 2,  $\xi' \in X_i$  such that  $\xi' \sqsubseteq \zeta$  and  $\mu_i(\xi') = p'$ . So  $\xi' \sqsubseteq \eta$  with  $\xi' \in X_i$ , whence  $\xi' = \xi$ , and so  $p' = p$ . Therefore, by Claim 1(vi),  $r = \mu_k(\zeta) \leq \mu_i(\xi') = p$ .  $\square$

Now we obtain the promised counterexample.

**Example 10.3.** *There exists a square (i.e., a diagram indexed by  $\mathbf{2}^2$ ) of finite distributive  $\langle \vee, 0 \rangle$ -semilattices and  $\langle \vee, 0, 1 \rangle$ -embeddings that does not have any simultaneous lattice embedding into any diagram of finite Boolean semilattices.*

*Proof.* Identify the finite poset  $P$  diagrammed on the left hand side of Figure 10.1 with its canonical image in the (distributive) lattice  $A$  of all ideals of  $P$ . So,  $1 = p_1 \vee p_2$  and  $P = J(A)$ . Put  $p = p_1 \wedge p_2 = q_1 \vee q_2$  and let  $S$  denote the  $\langle \vee, 0 \rangle$ -subsemilattice of  $A$  generated by  $\{p, p_1, p_2\}$ . Hence  $S$  is distributive and  $J(S) = \{p, p_1, p_2\}$ . For  $i \in \{1, 2\}$ , put  $r_i = q_i \vee q'_i$  and  $P_i = \{p, p_1, p_2, r_i\}$ . Denote by  $A_i$  the  $\langle \vee, 0 \rangle$ -subsemilattice of  $A$  generated by  $P_i$ . The only nontrivial comparable pairs in  $P_1$  are given by  $r_1 < p_1$  and  $p < p_1, p_2$ . Furthermore, since  $p_1 \not\leq r_1 \vee p_2$ , all elements of  $P_1$  are join-prime in  $A_1$ , hence  $A_1$  is isomorphic to the lattice of all ideals

FIGURE 10.1. The poset  $P$  and the  $\langle \vee, 0, 1 \rangle$ -semilattices  $S$  and  $A_1$ .

of  $P_1$ ; whence it is distributive. Similarly,  $A_2$  is distributive. The semilattices  $S$  and  $A_1$  are diagrammed on Figure 10.1. Of course,  $A_1$  and  $A_2$  are isomorphic.

So we have obtained four  $\langle \vee, 0, 1 \rangle$ -semilattices  $S \subseteq A_1, A_2 \subseteq A$ . Suppose that this square has a simultaneous lattice embedding into a diagram of finite Boolean semilattices. We apply Proposition 10.2 to the element  $p \in J(S)$ . The element  $q \in J(A)$  given by Proposition 10.2 lies below  $p$ , so we may assume, by symmetry, that  $q = q_1$ . From  $r_1 \in \partial^{A_1, A} q_1$  and  $r_1 \leq p_S$ , it follows that  $r_1 \leq p$ , a contradiction.  $\square$

The proof of Proposition 10.2 above makes essential use of the distributivity of all  $B_i$ -s. As we shall see in Section 11, this is unavoidable.

## 11. THE GRÄTZER-SCHMIDT EXTENSION AND RETRACTS OF ULTRA-SIMPLE-ATOMISTIC SEMILATTICES

A well-known result by M. Tischendorf [14] gives a direct construction implying that *every finite lattice embeds into some finite atomistic lattice*. For a finite lattice  $L$ , denote by  $\mathbf{Ti}(L)$  the finite atomistic lattice obtained from  $L$  via Tischendorf's construction. It is proved in [14] that  $\mathbf{Ti}(L)$  is a finite atomistic lattice containing (as a bounded lattice)  $L$ , via the embedding  $\zeta_L : L \hookrightarrow \mathbf{Ti}(L)$ ,  $a \mapsto \downarrow a \cap J(L)$ . In fact, it is proved in [14] that  $\zeta_L$  is *congruence-preserving*, that is, the natural map from the congruence lattice  $\text{Con } L$  of  $L$  to  $\text{Con } \mathbf{Ti}(L)$  is an isomorphism. Furthermore, the map  $\rho_L : \mathbf{Ti}(L) \rightarrow L$ ,  $X \mapsto \bigvee X$  is easily seen to be a  $\langle \vee, 0 \rangle$ -retraction of  $\zeta_L$ .

Although the correspondence  $L \mapsto \mathbf{Ti}(L)$  cannot be extended "naturally" to arbitrary  $\langle \vee, 0 \rangle$ -embeddings, it can be extended to *isomorphisms*. This is sufficient to construct from it an appropriate shelter. This shelter can, in turn, be used to prove the following analogue of Corollary 9.7: *Every  $\langle \vee, 0 \rangle$ -semilattice is a retract of a  $\langle \vee, 0 \rangle$ -semilattice which is a directed  $\langle \vee, 0 \rangle$ -union of finite atomistic lattices*.

However, a much stronger result can be proved with a much simpler method, see Theorem 11.5. We shall now present this proof.

We recall that a  $\langle \vee, 0 \rangle$ -semilattice  $K$  is *atomistic*, if every element of  $K$  is a join of atoms of  $K$ . The purpose of the first part of the following definition is to separate the two distinct notions of simple semilattice (which is a trivial) and simple lattice.

**Definition 11.1.** Let  $K$  be a lattice with zero. We say that the  $\langle \vee, 0 \rangle$ -semilattice  $\langle K, \vee, 0 \rangle$  is *lattice-simple*, if the lattice  $\langle K, \vee, \wedge \rangle$  is simple. A  $\langle \vee, 0 \rangle$ -semilattice  $K$  is *ultra-simple-atomistic*, if  $K$  is the directed union of its finite, lattice-simple, atomistic  $\langle \vee, 0 \rangle$ -subsemilattices.

We denote by  $\text{At } K$  the set of all atoms of a  $\langle \vee, 0 \rangle$ -semilattice  $K$ , and we put  $\text{NAt } K = K \setminus (\{0\} \cup \text{At } K)$ . For every  $a \in \text{NAt } K$ , we adjoin distinct atoms  $p_a^i < a$ , for  $i \in \{0, 1\}$ , such that  $p_a^i = p_b^j$  only in case  $a = b$  and  $i = j$ . Now we put

$$\mathbf{GS}(K) = K \cup \{p_a^i \mid a \in \text{NAt } K \text{ and } i < 2\}.$$

Since this construction is used in the proof of [7, Lemma 7], we shall call it the *Grätzer-Schmidt extension* of  $K$ .

The ordering of  $\mathbf{GS}(K)$  consists of the ordering of  $K$ , augmented by the following pairs:

$$\begin{aligned} p_a^i \leq b &\iff a \leq b && \text{(in case } a \in \text{NAt } K\text{),} \\ a \leq p_b^j &\iff a = 0 && \text{(in case } b \in \text{NAt } K\text{),} \\ p_a^i \leq p_b^j &\iff (a = b \text{ and } i = j) && \text{(in case } a, b \in \text{NAt } K\text{),} \end{aligned}$$

for  $a, b \in K$  and  $i, j < 2$ .

The following lemma records a few straightforward properties of  $\mathbf{GS}(K)$ .

**Lemma 11.2.** Let  $K$  be a  $\langle \vee, 0 \rangle$ -semilattice. Then the following properties hold:

- (i) The ordering  $\leq$  endows  $\mathbf{GS}(K)$  with a structure of  $\langle \vee, 0 \rangle$ -semilattice.
- (ii) The inclusion map  $\varepsilon_K: K \hookrightarrow \mathbf{GS}(K)$  is a complete  $\langle \vee, \wedge \rangle$ -embedding (that is, an order-embedding that preserves all meets and joins defined in  $K$ ).
- (iii) If  $K$  is a lattice, then so is  $\mathbf{GS}(K)$ .
- (iv) The map  $\mu_K: \mathbf{GS}(K) \rightarrow K$  extending  $\text{id}_K$  such that  $\mu_K(p_a^i) = a$ , for all  $a \in \text{NAt } K$  and  $i < 2$ , is a  $\langle \vee, 0 \rangle$ -homomorphism, and  $\mu_K \circ \varepsilon_K = \text{id}_K$ .

For elements  $a$ ,  $b$ , and  $c$  in a  $\langle \vee, 0 \rangle$ -semilattice  $K$ , we say that  $c = a \oplus b$ , if  $c = a \vee b$  and  $a \wedge b = 0$ . Moreover, we say that  $a$  and  $b$  are *perspective*, in notation  $a \sim b$ , if there exists  $x \in K$  such that  $a \oplus x = b \oplus x$ . The following lemma contains further related properties.

**Lemma 11.3.** Let  $K$  be a  $\langle \vee, 0 \rangle$ -semilattice. Then the following properties hold:

- (i) Every element of  $\mathbf{GS}(K)$  is a join of at most two atoms.
- (ii) For all  $x, y \in \mathbf{GS}(K)$  such that  $0 < x < y$ , there exists an atom  $p$  of  $\mathbf{GS}(K)$  such that  $y = x \oplus p$ .
- (iii) Any two atoms of  $\mathbf{GS}(K)$  are perspective.
- (iv) If  $K$  is a lattice, then  $\mathbf{GS}(K)$  is lattice-simple.
- (v) In the general case,  $\mathbf{GS}(K)$  is ultra-simple-atomistic.

*Proof.* (i) Any  $a \in \text{NAt } K$  satisfies that  $a = p_a^0 \vee p_a^1$ .

(ii) Necessarily,  $y \in K$ . If  $x \in K$ , then  $y = x \oplus p_y^0$ . If  $x = p_a^i$ , then  $a \leq y$ , and so  $p_a^i \oplus p_y^{1-i} = y$ .

(iii) Let  $x$  and  $y$  be distinct atoms of  $\mathbf{GS}(K)$ , and put  $c = \mu_K(x) \vee \mu_K(y)$ . If  $x, y \in \text{At } K$ , then  $x \oplus p_c^0 = y \oplus p_c^0 = c$ . If  $x \in \text{At } K$  and  $y = p_a^i$  (so  $\mu_K(y) = a$ ), then  $x \oplus p_c^{1-i} = y \oplus p_c^{1-i} = c$ . Suppose that  $x = p_a^i$  and  $y = p_b^j$ . Since  $a \in \text{NAt } K$ , there exists  $d \in K$  such that  $0 < d < a$ . If  $a = b$ , then  $x \oplus d = y \oplus d = a$ . Suppose that  $a \neq b$ , say  $b \not\leq a$ . Then  $x \oplus p_c^{1-j} = y \oplus p_c^{1-j} = c$ .

(iv) It follows from Lemma 11.2(iii) that  $\mathbf{GS}(K)$  is a lattice. Denote by  $\Theta(x, y)$  the (lattice-)congruence of  $\mathbf{GS}(K)$  generated by the pair  $\langle x, y \rangle$ , for any  $x, y \in \mathbf{GS}(K)$ . It follows from (iii) that  $\Theta(0, x) = \Theta(0, y)$ , for all atoms  $x$  and  $y$  of  $\mathbf{GS}(K)$ . Therefore, by (i) (or (ii)),  $\mathbf{GS}(K)$  is lattice-simple.

(v) As  $K$  is the directed union of its finite  $\langle \vee, 0 \rangle$ -subsemilattices (we define the empty directed union as  $\{0\}$ ), we obtain that  $\mathbf{GS}(K)$  is the directed union of all  $\mathbf{GS}(F)$ , for  $F$  a nontrivial finite join-subsemilattice of  $K$ . By (i) and (iv),  $\mathbf{GS}(F)$  is finite, atomistic, and lattice-simple, for all such  $F$ .  $\square$

For  $\langle \vee, 0 \rangle$ -semilattices  $K$  and  $L$  and a  $\langle \vee, 0 \rangle$ -embedding  $f: K \hookrightarrow L$ , we define a map  $\mathbf{GS}(f): \mathbf{GS}(K) \rightarrow \mathbf{GS}(L)$  by the rule

$$\begin{aligned} \mathbf{GS}(f)(a) &= a && \text{for } a \in K, \\ \mathbf{GS}(f)(p_a^i) &= p_{f(a)}^i && \text{for } a \in \text{NAt } K \text{ and } i < 2. \end{aligned}$$

The verification of the following lemma is straightforward.

**Lemma 11.4.** *In the context above, the map  $\mathbf{GS}(f)$  is a  $\langle \vee, 0 \rangle$ -embedding from  $\mathbf{GS}(K)$  into  $\mathbf{GS}(L)$  such that  $\mathbf{GS}(f) \circ \varepsilon_K = \varepsilon_L \circ f$  and  $\mu_L \circ \mathbf{GS}(f) = f \circ \mu_K$ . Furthermore, if  $f$  is a lattice homomorphism, then so is  $\mathbf{GS}(f)$ .*

Putting together some of the information above, we obtain the following rather elementary result.

**Theorem 11.5.** *The triple  $\langle \mathbf{GS}, \varepsilon, \mu \rangle$  is a functorial retraction of the category of  $\langle \vee, 0 \rangle$ -semilattices and  $\langle \vee, 0 \rangle$ -embeddings to the full subcategory of ultra-simple-atomistic  $\langle \vee, 0 \rangle$ -semilattices. Furthermore, the functor  $\mathbf{GS}$  sends finite lattices to finite (lattice-)simple lattices.*

The essence of this result can be captured by the following somewhat loose formulation: *Every  $\langle \vee, 0 \rangle$ -semilattice is a retract of some ultra-simple-atomistic  $\langle \vee, 0 \rangle$ -semilattice, and this holds functorially.*

Further properties of the functorial retraction of Theorem 11.5 are obtained above. For example, for any  $\langle \vee, 0 \rangle$ -embedding  $f: K \hookrightarrow L$ ,

- (1) the map  $\varepsilon_K$  is a complete  $\langle \vee, \wedge \rangle$ -embedding (this is why the assumption of distributivity of the  $B_i$ -s is unavoidable in the proof of Proposition 10.2);
- (2) if  $f: K \hookrightarrow L$  is a 0-lattice embedding, then so is  $\mathbf{GS}(f)$ .

Let us keep the notation of Section 9 for  $\mathcal{S}$  and  $\mathcal{M}$ , and denote by  $\mathcal{S}_{\text{at}}$  the full subcategory of atomistic members of  $\mathcal{S}$ . We state the following analogue of Proposition 9.6.

**Proposition 11.6.** *The projection functor from  $\text{Retr}(\mathcal{S} \cap \mathcal{M}, \mathcal{S}_{\text{at}} \cap \mathcal{M})$  to  $\mathcal{S} \cap \mathcal{M}$  has neither a right nor a left adjoint.*

The proof of Proposition 11.6 is virtually the same as the one of Proposition 9.6.

However, as shows the following easy result and since there are finite non-atomistic lattices, the analogue of Theorem 11.5 for *lattices* does not hold.

**Proposition 11.7.** *Any finite  $\langle \vee, \wedge \rangle$ -homomorphic image of a  $\langle \vee, \wedge \rangle$ -direct limit of finite atomistic lattices is atomistic.*

*Proof.* Let  $K$  be a finite lattice and let  $g: L \twoheadrightarrow K$  be a surjective lattice homomorphism, where  $L = \varinjlim_{i \in I} L_i$ , with  $I$  directed, the lattices  $L_i$  finite atomistic, and transition maps  $f_i: L_i \rightarrow L$ . Since  $K$  is finite, there exists  $i \in I$  such that  $g_i = g \circ f_i$  is surjective. Since  $g_i(p)$  is an atom of  $K$ , for any atom  $p$  of  $L_i$ ,  $K$  is atomistic.  $\square$

## 12. OPEN PROBLEMS

As observed above, the functorial retraction constructed in the proof of Theorem 9.5, although theoretically computable, lives *a priori* beyond the reach of any implementation. The most natural question is thus whether such a functorial retraction could be constructed with ‘reasonable’ growth.

A possible way to formulate this problem is the following. We use the notation of Section 9.

**Problem 1.** Are there a functorial retraction  $\langle \Phi, \varepsilon, \mu \rangle$  of  $\mathcal{D} \cap \mathcal{M}$  to  $\mathcal{B} \cap \mathcal{M}$  such that  $|J(\Phi(D))|$  is bounded by a polynomial in  $|J(D)|$ , for every finite distributive (semi)lattice  $D$ ?

Both Example 10.3 and the huge upper bound for the construction of Theorem 9.5 suggest that  $\Phi(D)$  needs to be large with respect to  $D$ .

Say that a  $\langle \vee, 0 \rangle$ -homomorphism  $\mu: S \rightarrow T$  is *weakly distributive*, if whenever  $\mu(c) = a \vee b$ , there is a decomposition  $c = x \vee y$  in  $S$  such that  $\mu(x) \leq a$  and  $\mu(y) \leq b$ . In view of some lifting results with respect to the congruence functor on lattices (see [16] for a survey), the following problem may be relevant.

**Problem 2.** Is every distributive  $\langle \vee, 0 \rangle$ -semilattice a weakly distributive image of some ultraboolean  $\langle \vee, 0 \rangle$ -semilattice?

We know that Problem 2 has a positive answer for *countable* semilattices.

We do not know whether the analogue of Corollary 9.7 for *dimension groups* holds. By definition, a partially ordered abelian group  $G$  is a *dimension group*, if  $G$  is directed (for its ordering), unperforated, and has the interpolation property, see [4]. Special cases of dimension groups are the *simplicial groups*, that is, those partially ordered abelian groups that are isomorphic to some finite power of the integers (with componentwise ordering). As defined in [3], a partially ordered abelian group is **E-ultrasimplicial**, if it is a directed union of simplicial groups. Every **E-ultrasimplicial** group is a dimension group; the converse is easily seen to be false, even in the divisible case (see [3, Example 1.2]).

**Problem 3.** Is every dimension group a retract of some **E-ultrasimplicial** group?

A similar question can be formulated in the context of [5]. We denote by  $\mathcal{R}_{\text{ep}}$  (resp.,  $\mathcal{R}_{\text{ep}}^*$ ) the class of all monoids which are direct limits (resp., directed unions) of finite products of monoids of the form  $(\mathbb{Z}/n\mathbb{Z}) \cup \{0\}$  for positive integers  $n$ . A first-order characterization of  $\mathcal{R}_{\text{ep}}$  is obtained in [5].

**Problem 4.** Is every member of  $\mathcal{R}_{\text{ep}}$  a retract of some member of  $\mathcal{R}_{\text{ep}}^*$ ?

The result of Theorem 9.5 is made possible by the shelter **B**. In order to define a shelter we need a functor playing the role of the ‘functor from  $\mathcal{S}^{\text{iso}}$  to  $\mathcal{B}^{\text{iso}}$ ’ of Definition 4.2(i). A special feature of such functors is that they can be easily defined on *isomorphisms* (because they are given by ‘explicit’ constructions), but not on *embeddings*. There are probably many such objects within mathematical practice. For example, it is proved in [1, Theorem 1.11], *via* an explicit construction, that every finite join-semidistributive lattice embeds into some finite atomistic join-semidistributive lattice. (A lattice is *join-semidistributive*, if it satisfies the quasi-identity  $x \vee y = x \vee z \Rightarrow x \vee y = x \vee (y \wedge z)$ .) This suggests the following problem.

**Problem 5.** Say that a  $\langle \vee, 0 \rangle$ -semilattice  $S$  is *join-semidistributive*, if for all  $a, b, c \in S$ , if  $a \vee b = a \vee c$ , then there exists  $x \leq b, c$  such that  $a \vee b = a \vee x$ . Is every join-semidistributive  $\langle \vee, 0 \rangle$ -semilattice a retract of some direct limit of finite atomistic join-semidistributive  $\langle \vee, 0 \rangle$ -semilattices?

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